PROPERTIES OF ORDERED RANDOM SCHEMES

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

ZUBER AKHTER

UNDER THE SUPERVISION OF

Dr. HASEEB ATHAR

DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202 002, INDIA

2016

Department of Statistics & Operations Research
Aligarh Muslim University
Aligarh – 202002, India

Candidate's Declaration

I, **Zuber Akhter**, a Research Student of the Department of Statistics & Operations Research, Aligarh Muslim University, Aligarh certify that the work embodied in this Ph.D. thesis is my own bonafide work carried out by me under the supervision of **Dr. Haseeb Athar** at the Department of Statistics & Operations Research, Aligarh Muslim University, Aligarh. The matter embodied in this Ph.D. thesis has not been submitted for the award of any other degree.

I declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and in the body of the thesis. I further certify that I have not willfully lifted up some other's work, para, text, data, result, etc. reported in the journals, books, magazines, reports, dissertations, theses, etc., or available at websites and included them in this Ph.D. thesis and cited as my own work.

Dated ………….. (Zuber Akhter)

Certificate from the Supervisor

This is to certify that the above statement made by the candidate is correct to the best of my knowledge and belief.

Signature of the Supervisor: ....

Name and Designation: Dr. Haseeb Athar, Assistant Professor

Department: Statistics & Operations Research,
Aligarh Muslim University, Aligarh

(Signature of the Chairman of the Department with seal)

Phone: +91 571 2701251 (Office), e-mail: haseebathar@hotmail.com

Department of Statistics & Operations Research
Aligarh Muslim University
Aligarh – 202002, India

Qazi Mazhar Ali
Professor & Chairman

Course/Comprehensive Examination/Pre-submission Seminar Completion Certificate

This is to certify that ***Mr. Zuber Akhter****, Department of Statistics & Operations Research has satisfactorily completed the course work/comprehensive examination and pre-submission seminar requirement which is part of his Ph.D. (Statistics) programme.*

Dated :

(Prof. Qazi Mazhar Ali)
Chairman

Phone: +91 571 2701251 (Office), e-mail: chairman.stats@gmail.com

Department of Statistics & Operations Research
Aligarh Muslim University
Aligarh – 202002, India

Copyright Transfer Certificate

Title of the Thesis : **Properties of Ordered Random Schemes**

Candidate's Name : **Zuber Akhter**

Copyright Transfer

The undersigned hereby assigns to the Aligarh Muslim University, Aligarh copyright that may exist in and for the above thesis submitted for the award of the Ph.D. degree.

(Zuber Akhter)
(Signature of the Candidate)

Note: However, the author may reproduce or authorize others to reproduce material extracted verbatim from the thesis or derivative of the thesis for author's personal use provided that the source and the University's copyright notice are indicated.

LIST OF PUBLICATIONS

1. Athar, H. and Akhter, Z. (2016): Some results on relations between moments of generalized order statistics. *J. Adv. Res. Appl. Math.* In press.
2. Noor, Z., Akhter, Z. and Athar, H. (2015): On characterization of probability distributions through conditional expectation of generalized and dual generalized order statistics. *Pak. J. Statist.*, **31**(2), 159-170.
3. Athar, H. and Akhter, Z. (2015a): Recurrence relations between moments of progressive Type-II right censored order statistics from doubly truncated Weibull distribution. *Statist. Papers*, DOI: 10.1007/s00362-015-0658-2.
4. Athar, H. and Akhter, Z. (2015b): Some characterization of continuous distributions based on order statistics. *Int. J. Comp. Theo. Stat.*, **2**(1), 31-36.
5. Athar, H., Akhter, Z. and Saran, J. (2014): Moments of progressive Type-II right censored order statistics from Lindley distribution. *Statistics Research Letters*, **3**(1), 1-6.
6. Khan, M.J.S., Akhter, Z. and Wahid, N. (2016): Characterization of continuous probability distributions conditioned on a pair of record statistics. *ProbStat. Forum*. Under revision.

Dedicated

to my beloved Parents, especially to my father

Late Mohammad Hadis

CONTENTS

ACKNOWLEDGEMENTS

All praises due to the Almighty **Allah**, the most Beneficent, the most Merciful, Who bestowed upon me the courage, patience and strength to embark upon this work and carry it to its completion and admires to His last and great Prophet, **Muhammad (PBUH)**, whose moments of life provided me with the strength to run on the path of success.

Most humbly, I would like to express my heartfelt gratitude and indebtedness to my reverend supervisor Dr. Haseeb Athar, for his invaluable suggestions, continuous support, superb guidance and consistent encouragement throughout the research period. He will always be a source of inspiration for me and I feel myself really lucky to have such a great supportive supervisor. It is only his personal influence, expert guidance and boundless support that enabled me to complete the work in the present form.

I owe the deepest appreciation and gratitude to Prof. Abdul Hamid Khan, former Dean, Faculty of Science and Ex-Chairman, Department of Statistics & Operations Research, Aligarh Muslim University, Aligarh. I am extremely grateful to him for holding me to a high research standard and enforcing strict validations for each result and thus teaching me how to do research. I will forever be grateful to him for his kindness as a teacher.

I am also thankful to Prof. Qazi Mazhar Ali, Chairman, Department of Statistics & Operations Research, Aligarh Muslim University, Aligarh, for his cooperation, encouragement and providing all necessary facilities to accomplish this work within the stipulated time.

I would like to express my sincere gratitude and thanks to Dr. Rafiqullah Khan for their moral support and wholehearted cooperation.

My sincere thanks are extended to all other teachers of the department for their inspiration, encouragement and generous help.

It is my pleasure to express my deep sense of appreciation to all my seniors, juniors and colleagues, especially Dr. Saba Khalid Khwaja, Dr. Nayabuddin, Ms. Zubdah-e-Noor and Mr. Mohammad Azam Khan for their cooperation and sympathetic behaviour at all stages of this work. Sincere thanks to my friends and of course my roommates for their help in carrying out this work. I am also thankful to those whom I failed to mention.

Thanks are also due to all non-teaching staff members and all the seminar library staffs of the Department for their help and cooperation.

No words can actually express my deepest thanks and gratitude to my beloved parents who stood behind me in all the sweet and sour moments of life and were always there to pat my shoulder affectionately. They encouraged and helped me at every walk of my life. Their unconditional love, patience and sacrifice are vested in every page of this work. Whenever I felt tired and sense of despair gripped me, I always found them standing by my side, providing me words of immense consolation. I am forever indebted for their everlasting love and endless prayer.

And last but not the least I would also like to express my special thanks to my brothers, sisters and brothers-in-law for their prayer, moral support and always encouraging me with their best wishes. Special acknowledgement goes to my lovely nephews Shaad, Rehan, Shahid, and Shadab for their love and dua.

I am also thankful to University Grant Commission, New Delhi for providing the financial assistance during my research period.

Dated: *Zuber Akhter*

PREFACE

The theory of ordered random variables received a tremendous attention from many researchers during the past century. This theory deals with the properties and applications of ordered random variables and functions involving them. There are several models of ordered random variables such as order statistics, record values, sequential order statistics, progressively Type-II censored order statistics and generalized order statistics with interesting applications in many fields of statistical science.

The independent and identically distributed random variables, when put in ascending order magnitude are called order statistics. Order statistics appear in several areas of studies such as detection of outlier, robustness, quality control, reliability analysis and so on. Further it is also involved in statistical modelling, statistical inferences, decision procedures, nonparametric statistics, among others. The distribution theory of order statistics has been widely described in several monographs written by outstanding statisticians and there are numerous papers devoted to the theory of order statistics and its applications [See Arnold *et al*. (1992), David and Nagaraja (2003].

Chandler (1952) introduced the record values and documented many of the basic properties of records. An observation is called a record if its value is greater than (or less than) all the previous observations. Record values have found important applications in many fields of studies including sports, science, engineering, medicine, climatology, traffic, industry, among others. For more detail survey on this topic one may refer to the books by Arnold *et al.* (1998), Nevzorov (2001), Ahsanullah (1995) and references therein.

The progressively Type-II censored order statistics is an interesting generalization of order statistics, which is very useful in reliability and lifetime studies. Several fundamental results in this interesting and attractive topic are available in the books by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

Kamps (1995) introduced the model of generalized order statistics and shown that several well known models of ascending ordered random variables are contained in this model of generalized order statistics in the distributional and theoretical sense [Bairamov, 2007].

Although, generalized order statistics contain many useful models of ascending order random variables. However, the random variables that are in descending order cannot be integrated into this framework. Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics that enables a common approach to study the descending order random variables like reverse order statistics and lower record values.

During the past centuries, order statistics and related general models of ordered random variables have played immense role in problem of deriving the recurrence relations and identities for single and product moments for some specific as well as general classes of distributions. These moments are widely used in statistical inference. Further, these models have also been utilized in the characterization problems. A vast literature reviews have been given in the subsequent chapters.

Recurrence relations and identities are important in their own right because they express the higher order moments in terms of the lower order moments and hence make the evaluation of higher order moments easy as well as reduce the time and labour.

The only method that enables the determination of a probability distribution exactly is a characterization theorem and therefore the study of such theorems has emerged as an important area of mathematical statistics. The characterization of probability distributions is mainly useful in the area of goodness of fit tests, which is helpful in the construction of statistical tests.

In this thesis, properties of ordered random variables are utilized to obtain the recurrence relations satisfied by the single and the product moments of ordered random variables from truncated and non-truncated continuous distributions.

Further, we have also exploited the conditional expectation of ordered random variables to characterize some general classes of distributions.

The thesis entitled **"Properties of Ordered Random Schemes"** is based on seven chapters, which are described as below:

Chapter I is introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

Chapter II is based on recurrence relations for single and product moments of progressively Type-II right censored order statistics from the doubly truncated Weibull distribution.

Chapter III contains the results based on recurrence relations for single and product moments of progressively Type-II right censored order statistics from Lindley distribution. Further, computational algorithm to compute moments is also given.

Chapter IV contains the results on recurrence relations for the moments of generalized order statistics when the continuous distribution function $F(x)$ and the probability density function $f(x)$ are functionally related as

$$f(x) = a\,x^b[1-F(x)]^c,\ x \in (\alpha,\beta)$$

where a, b and c are integers. Results for various distributions are given by properly choosing the parameters a, b and c.

Chapter V deals with the characterization of a general form of distribution $F(x) = ah(x)+b,\ x \in (\alpha,\beta)$ through conditional expectation of $p-th$ power of difference of functions of two order statistics, conditioned on a pair of non-adjacent order statistics. Examples of various distributions are given by properly choosing a, b and $h(x)$.

Chapter VI embodies the characterization of family of continuous distributions of the form $F(x) = 1-e^{-ah(x)},\ x \in (\alpha,\beta)$ through conditional expectation of $p-th$ power of difference of functions of two upper record values, conditioned on a pair of non-adjacent record values.

Chapter VII contains the characterization of some general form of continuous probability distributions based on generalized and dual generalized order statistics when conditioning is not adjacent. Further, deductions for order statistics and record values are also discussed and several examples are listed.

At the end, a comprehensive bibliography is given which has been referred in the thesis.

CHAPTER I

PRELIMINARIES

1. INTRODUCTION

Models of ordered random variables are important and are widely used in statistical theory and its applications. These models are order statistics, record values, progressively censored order statistics and generalized order statistics with different interpretations and have found interesting applications in many fields including reliability theory, survival analysis, financial economics etc.

In this chapter a brief review of the concepts and results used in subsequent chapters have been presented.

2. ORDER STATISTICS

The term “Order statistics” was first used by Wilks (1942). The extensive role of order statistics in several areas of statistical inference has made it imperative and useful to gather these results and present them in varied manner to suit diverse interests. Order statistics deals with the properties and applications of ordered random variables and of functions involving them. It utilizes the rank or order of an observation as well as its magnitude and combines the techniques of conventional statistics (which consider the magnitude of the observation) with those of rank order statistics (which consider only the relative rank whether or not the original observations were measured on an ordinal scale).

Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). David and Nagaraja (2003) is the basic book on order statistics dealing in detail with its different aspects. One may also refer to Sarhan and Greenberg (1962), Arnold and Balakrishnan (1989), Balakrishnan and Cohen (1991) and Arnold *et al.* (1992) for an elaborate reviews and developments of order statistics.

2.1 Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$. If these are arranged in ascending order of magnitude such that $X_{1:n} \le X_{2:n} \le \ldots \le X_{r:n} \le \ldots \le X_{n:n}$, then $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are collectively called the order statistics of the sample of size n and $X_{r:n}$ $(r = 1, 2, \ldots, n)$ is called the $r-th$ order statistic of the sample. Here $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are called extreme order statistics or the smallest and the largest order statistics respectively.

2.2 Applications

Order statistics play a vital role in the study of many natural problems related to flood, breaking strength, atmospheric temperature, atmospheric pressure, wind etc.

Order statistics can also be applied in instances in which short-cut or time-saving devices are appropriate for problems of estimation and/or tests of significance. For example, use of the median to estimate central tendency, use of the range to estimate dispersion, or an entire short-cut analysis of variance as suggested by Hartley (1942).

Order statistics and functions of order statistics play a very important role in statistical theory and methodology. There are many cases in which methods based on order statistics are the most efficient. In other cases, they are used because of their simplicity or their robustness, even at the cost of some loss of efficiency. For example, the sample mean and standard deviation provide efficient estimators of the corresponding population parameters under the assumption of normality, but the sample range is simpler to use than the sample median in statistical quality control, and the sample median furnish more robust estimators when the population may have longer tails than the normal. Extreme (largest and smallest) values are important in hydrology (floods and droughts), aeronautics (gust loads), oceanography (waves and tides), material strength ("weakest link" theory) and meteorology, duration of life on humans, other organisms and devices of various kinds. Order statistics occur naturally in life testing [Wilks (1948); Greenberg and Sarhan (1958)].

Besides the above applications of order statistics and other related statistics have also involved in statistical modeling, inferences, decision procedures, study of the reliability systems, robustness studies, statistical quality control, filtering theory, signal processing, image processing and radar target detection.

2.3 Distribution of order statistics

In this section, we will describe the basic distribution theory of order statistics for continuous population.

The pdf of $X_{r:n}$, $1 \le r \le n$ the $r-th$ order statistic is given by [David and Nagaraja (2003)]

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \ , \ -\infty < x < \infty . \tag{2.1}$$

The $pdf's$ of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x), \ -\infty < x < \infty , \tag{2.2}$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x), \ -\infty < x < \infty . \tag{2.3}$$

The $k-th$ moment of $X_{r:n}$ is defined as

$$\mu_{r:n}^{(k)} = E[X_{r:n}^{k}] = \int_{-\infty}^{\infty} x^k f_{r:n}(x)\, dx \tag{2.4}$$

and the joint pdf of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\times[1-F(y)]^{n-s} f(x) f(y), \quad x < y . \tag{2.5}$$

The $j-th$ and the $k-th$ product moment of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is given as

$$\mu_{r,s:n}^{(j,k)} = E[X_{r:n}^{j} X_{s:n}^{k}] = \iint\limits_{-\infty<x<y<\infty} x^j y^k f_{r,s:n}(x,y)\,dx\,dy\,. \tag{2.6}$$

The joint *pdf* of $X_{r:n}$, $X_{i:n}$ and $X_{s:n}$, $1 \le r < i < s \le n$ is given by

$$f_{r,i,s:n}(x,t,y) = \frac{n!}{(r-1)!(i-r-1)!(s-i-1)!(n-s)!}[F(x)]^{r-1}[F(t)-F(x)]^{i-r-1}$$

$$\times [F(y)-F(t)]^{s-i-1}[1-F(y)]^{n-s} f(x) f(t) f(y),$$

$$x < t < y\,. \tag{2.7}$$

In general, the joint *pdf* of $X_{i_1:n}, X_{i_2:n}, ..., X_{i_k:n}$ for $1 \le i_1 < i_2 < \cdots < i_k \le n$ is given by

$$f_{i_1,i_2,...,i_k:n}(x_{i_1:n}, x_{i_2:n}, ..., x_{i_k:n}) = n!\left\{\prod_{j=1}^{k} f(x_{i_j})\right\}\prod_{j=0}^{k}\left\{\frac{[F(x_{i_{j+1}})-F(x_{i_j})]^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!}\right\},$$

$$x_{i_1} < x_{i_2} < \cdots < x_{i_k} \tag{2.8}$$

where $x_0 = -\infty$, $x_{k+1} = +\infty$, $i_0 = 0$, $i_{k+1} = n+1$.

Properties:

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.
2. Regarding the probability integral transformation, if $X_{r:n}, 1 \le r \le n$, is the order statistic from a continuous distribution function (*df*) $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the $r-th$ order statistic from a uniform distribution on $U(0,1)$.
3. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (*iid*) random variables from a continuous distribution, then, the set of order statistics $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

5. Let X be a continuous random variable with $E[X_{r:n}] = \mu_{r:n}$,

   a) If $\mu = E(X)$ exits then $\mu_{r:n}$ exists, but converse is not necessarily true. That is, $\mu_{r:n}$ may exist for certain (but not all) values of r, even though μ may not exist.

   b) $\mu_{r:n}$ for all n determine the distribution completely.

2.4 Truncated and conditional distribution

Let X be a continuous random variable having the *pdf* $f(x)$ and the *df* $F(x)$ in the interval $[-\infty, \infty]$.

$$\text{Let} \quad \int_{-\infty}^{Q_1} f(x)dx = Q \quad \text{and} \quad \int_{-\infty}^{P_1} f(x)dx = P, \tag{2.9}$$

where Q_1 and P_1 are known constants. Then doubly truncated *pdf* of X is given by

$$\frac{f(x)}{P-Q}, \; x \in (Q_1, P_1) \tag{2.10}$$

and the corresponding *df* is given by

$$\frac{F(x)-Q}{P-Q}, \; x \in (Q_1, P_1). \tag{2.11}$$

The lower and upper truncation points are Q_1, P_1 respectively; the degrees of truncation are Q (from below) and $1-P$ (from above). If we put $Q = 0$, the distribution will be truncated to the right. Similarly, for $P = 1$, the distribution will be truncated to the left. Whereas for $Q = 0$, $P = 1$ we get the non-truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

2.5 Some important results

Result 1 (David and Nagaraja, 2003): The conditional distribution of $X_{s:n}$ given $X_{r:n} = x$ for $r < s$, is same as the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x, that is

$$f_{s|r}(y \mid x) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}} f(y),\ x \le y$$

$$= \frac{(n-r)!}{(s-r-1)!(n-s)!}\left[\frac{F(y)-F(x)}{1-F(x)}\right]^{s-r-1}\left[1-\frac{F(y)-F(x)}{1-F(x)}\right]^{n-s}\frac{f(y)}{1-F(x)}. \tag{2.12}$$

Result 2 (David and Nagaraja, 2003): The conditional distribution of $X_{r:n}$ given $X_{s:n} = y$ for $r < s$, is same as the distribution of the $r-th$ order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y, is given as

$$f_{r|s}(x \mid y) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}}{[F(y)]^{s-1}} f(x),\ \ x \le y$$

$$= \frac{(s-1)!}{(r-1)!(s-r-1)!}\left[\frac{F(x)}{F(y)}\right]^{r-1}\left[1-\frac{F(x)}{F(y)}\right]^{s-r-1}\frac{f(x)}{F(y)}. \tag{2.13}$$

Result 3 (David and Nagaraja, 2003): The conditional *pdf* of $X_{i:n}$ given $X_{r:n} = x$ and $X_{s:n} = y$ for $1 \le r < i < s \le n$ is

$$f_{i|r,s}(t \mid x, y) = \frac{(s-r-1)!}{(i-r-1)!(s-i-1)!} \frac{[F(t)-F(x)]^{i-r-1}[F(y)-F(t)]^{s-i-1}}{[F(y)-F(x)]^{s-r-1}} f(t)$$

$$= \frac{(s-r-1)!}{(i-r-1)!(s-i-1)!}\left[\frac{F(t)-F(x)}{F(y)-F(x)}\right]^{i-r-1}$$

$$\times\left[1-\frac{F(t)-F(x)}{F(y)-F(x)}\right]^{s-i-1}\frac{f(t)}{F(y)-F(x)},\ \ x < t < y, \tag{2.14}$$

which is the distribution of the $(i-r)-th$ order statistics in a sample of size $(s-r-1)$ drawn from $\dfrac{f(t)}{F(y)-F(x)}$ $(x<t<y)$, *i.e.* from the parent population truncated on the left at x and on the right at y.

Result 4: Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$f(X_{k:n} \mid X_{1:n}=x_1,\ldots,X_{r:n}=x_r,\ldots,X_{s:n}=x_s,\ldots,X_{n:n}=x_n)$$

$$= f(X_{k:n} \mid X_{r:n}=x_r,\ X_{s:n}=x_s),\quad r<k<s\,.$$

Therefore, conditioning on more than two order statistics is unnecessary in view of Markovian property of order statistics from continuous random variables.

3. PROGRESSIVELY TYPE-II CENSORED ORDER STATISTICS

There are many practical situations in life testing and reliability experiments in which items are lost or eliminated from experimentation before failure. Life tests and reliability experiments are usually conducted to obtain information regarding the quality of a product. Such experiments are usually much time consuming and due to which these are expensive. Further, in some situations it is neither possible and nor desirable to use complete sample. Therefore in such situations sample is made to censored and only a portion of sample of individuals is studied. A sample is said to be censored if out of n items placed on a life-test, only $m(<n)$ of them are actually observed to fail.

Censoring arises in various ways in the areas such as reliability and survival analysis and is unavoidable [*cf.* Meeker and Escobar (1998) and Lawless (2003)]. Censoring is a statistical tool which helps us to reduce the sample size and save the time of collecting sample which is used to estimate the population parameter.

There are several types of censoring schemes, in which Type-I and Type-II censoring schemes are the two well known censoring schemes. In Type-I censoring scheme, the duration of the life test is fixed and the number of failures is random, while in Type-II censoring scheme, the duration of the life test is random and the number of failures is fixed. The progressive Type-II censored sampling scheme is the natural extension of Type-II censoring scheme and is used when the loss or removal of units before failures may be unavoidable. Usually in this scheme the randomly selected and removed units are all predetermined. However, in some practical situations the number of removal units may occur at random.

Progressive Type-II censored samples have been considered previously by Herd (1956, 1957, 1960), Roberts (1962a, 1962b), Cohen (1963), among others. Later Cohen (1991), Balakrishnan and Cohen (1991) and Balakrishnan and Aggarwala (2000) have summarized all these developments.

3.1 Definition and distribution

Consider an experiment in which n independent and identical items are put on life-test with continuous identically distributed failure times. A censoring schemes of R_i's is a set of prefixed integers such that at the time of the first failure, R_1 surviving items are removed from the experiment at random from the remaining $n-1$ surviving items. At the time of second observed failure, R_2 surviving items are removed from the experiment at random from the remaining $n-2-R_1$ surviving items. The process continuous until the $m-$th failure time at which, all the remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving items are removed from the experiment. We shall denote the m ordered observed failure times by $X_{1:m:n}^{\tilde{R}},\ldots,X_{m:m:n}^{\tilde{R}}$, and call them the progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $\tilde{R}=(R_1,R_2,\ldots,R_m)$, $m \le n$. If the failure times of the n items originally on test are from continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, then the joint *pdf* of $X_{1:m:n}^{\tilde{R}},\ldots,X_{m:m:n}^{\tilde{R}}$ is given by [Balakrishnan and Sandhu (1995), Balakrishnan *et al.* (2001) and Balakrishnan (2007)].

$$f^{X^{\tilde{R}}_{1:m:n},\ldots,X^{\tilde{R}}_{m:m:n}}(x_1,\ldots,x_m)=c(n,m-1)\prod_{j=1}^{m} f(x_j)[1-F(x_j)]^{R_j},$$

$$-\infty < x_1 < x_2 < \cdots < x_m < \infty, \qquad (3.1)$$

where,

$$n = m+\sum_{j=1}^{m} R_j,\ \ n,m\in N,\ \ R_j = N_0,\ 1\le j\le m,\ \ \tilde{R}=(R_1,R_2,\ldots,R_m),$$

and

$$c(n,m-1)=\prod_{j=1}^{m}\left(n-\sum_{i=1}^{j-1} R_i - j+1\right)=\prod_{j=1}^{m} R_j^{*} \qquad (3.2)$$

with $R_j^{*}=\sum_{k=j}^{m}(R_k+1)$. Observe that $R_1^{*}=n$.

The model of ordinary order statistics is contained in the above set up by choosing $\tilde{R}=(0,\ldots,0)$ (*i.e.* $m=n$) as censoring schemes, where no withdrawals are made.

3.2 Applications

The progressive Type-II censored sampling scheme is a versatile censoring scheme because it allows for items to be removed before the termination of the experiment to save time and cost of the life-testing experiment. Further, this scheme is especially useful when life tests involve expensive items.

The progressively Type-II censored order statistics is an interesting generalization of order statistics, which is quite useful in reliability and lifetime studies. For an overview of various developments concerning the distributional properties and the possible applications of progressively Type-II censored order statistics, one may refer to books by Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014) and review article by Balakrishnan (2007).

4. RECORD VALUES AND RECORD TIMES

The concept of record values was first given by Chandler (1952). Record values are defined as a model of successive extremes in a sequence of *iid* random variables [*cf.* Glick (1978)]. In other words, record values are simply referred to as the smallest (largest) observation among all the previously recorded values. The theory of record values depends largely on the theory of order statistics and is especially closely connected to extreme order statistics.

For major developments on theory, methods and applications of record values one may refer to Nagaraja (1988a), Ahsanullah (1995), Arnold *et al.* (1998) and Nevzorov (2001).

4.1 Definition

Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables with the *df* $F(x)$ and the *pdf* $f(x)$. Let $Y_n =$ max(min)$\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper (lower) record values of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as lower record value. One can transform the upper record values to lower record values by replacing the original sequence of $\{X_j, j \geq 1\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_i > 0) = 1$ for all i) by $\left\{\frac{1}{X_i}, i \geq 1\right\}$, the lower record value of this sequence will correspond to the upper record values of the original sequence [Ahsanullah (1995)].

The indices at which the upper record values occur are given by the record times $\{U(n), n > 0\}$. That is $X_{U(n)}$ is the $n-th$ upper record value, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. The distribution of $U(n), n \geq 1$ does not depend on F. Further, we will denote $L(n)$ as the indices where the lower record values occur. By assumption $U(1) = L(1) = 1$. The distribution of $L(n)$ also does not depend on F.

4.2 Applications

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recoding them, for example, Olympic records or world records in sports.

Record values arise naturally in many fields of studies such as Industrial stress testing, engineering, meteorological analysis, oil and mining surveys, sports, science, medicine, climatology, traffic, industry and so on. Further, it is of interest to note that there are many situations in which only records are observed, such as, in meteorology, hydrology, seismology and mining.

Records are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. In reliability theory, order statistics and record values are used for statistical modeling.

4.3 Distribution of record values

Let $F_{U(r)}(x)$ be the *df* of $X_{U(r)}$, $r \geq 1$, then we have [Ahsanullah (1995)]

$$F_{U(r)}(x) = P(X_{U(r)} \leq x) = \int_{-\infty}^{x} \frac{[-\ln \overline{F}(u)]^{r-1}}{(r-1)!} dFu\,, \quad -\infty < x < \infty \tag{4.1}$$

where $\overline{F}(x) = 1 - F(x)$, $0 < \overline{F}(x) < 1$ and 'ln' is the natural logarithm.

The *pdf* $f_{U(r)}(x)$ of $X_{U(r)}$ is

$$f_{U(r)}(x) = \frac{[-\ln \overline{F}(x)]^{r-1}}{(r-1)!} f(x), \quad -\infty < x < \infty \tag{4.2}$$

and the joint *pdf* of $X_{U(r)}$ and $X_{U(s)}$, $r < s$ is

$$f_{U(r),U(s)}(x,y) = \frac{1}{(r-1)!(s-r-1)!} [-\ln \overline{F}(x)]^{r-1}$$
$$\times [-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \frac{f(x)}{\overline{F}(x)} f(y), \quad x < y. \tag{4.3}$$

Similarly, the joint *pdf* of $X_{U(r)}$, $X_{U(i)}$ and $X_{U(s)}$, $1 \leq r < i < s$ is given by

$$f_{U(r),U(i),U(s)}(x,t,y)=\frac{1}{(r-1)!(i-r-1)!(s-i-1)!}[-\ln\overline{F}(x)]^{r-1}$$

$$\times[-\ln\overline{F}(t)+\ln\overline{F}(x)]^{i-r-1}[-\ln\overline{F}(y)+\ln\overline{F}(t)]^{s-i-1}$$

$$\times\frac{f(x)}{\overline{F}(x)}\frac{f(t)}{\overline{F}(t)}f(y),\quad x<t<y. \tag{4.4}$$

Thus, the conditional *pdf* of $X_{U(s)}$ given $X_{U(r)}=x$ is

$$f_{U(s)|U(r)}(y\,|\,x)=\frac{[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}}{(s-r-1)!}\frac{f(y)}{\overline{F}(x)},\quad x<y \tag{4.5}$$

and the conditional *pdf* of $X_{U(i)}$ given $X_{U(r)}=x$ and $X_{U(s)}=y$ is

$$f_{U(i)|U(r),U(s)}(t\,|\,x,y)=\frac{(s-r-1)!}{(i-r-1)!(s-i-1)!}\frac{[-\ln\overline{F}(t)+\ln\overline{F}(x)]^{i-r-1}}{[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}}$$

$$\times[-\ln\overline{F}(y)+\ln\overline{F}(t)]^{s-i-1}\frac{f(t)}{\overline{F}(t)},\quad x<t<y. \tag{4.6}$$

The *pdf* of the $r-th$ lower record value $X_{L(r)}$ can be obtained by replacing '$-\ln\overline{F}(x)$' with '$-\ln F(x)$', $0<F(x)<1$ in the *pdf* of $r-th$ upper record value $X_{U(r)}$. Therefore,

$$f_{L(r)}(x)=\frac{[-\ln F(x)]^{r-1}}{(r-1)!}f(x),\quad -\infty<x<\infty \tag{4.7}$$

and the joint *pdf* of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{L(r),L(s)}(x,y)=\frac{1}{(r-1)!(s-r-1)!}[-\ln F(x)]^{r-1}$$

$$\times[-\ln F(y)+\ln F(x)]^{s-r-1}\frac{f(x)}{F(x)}f(y),\quad x>y. \tag{4.8}$$

The conditional *pdf* of $X_{L(s)}$ given $X_{L(r)}=x$ is

$$f_{L(s)|L(r)}(y\,|\,x)=\frac{[-\ln F(y)+\ln F(x)]^{s-r-1}}{(s-r-1)!}\frac{f(y)}{F(x)},\quad x>y \tag{4.9}$$

4.4 k-Records

In some situations record values themselves are viewed as 'outlier' and hence second or third largest values are of special interest, then the $k-th$ record values proposed by Dziubdziela and Kopociński (1976) is adequate. Insurance claims in some non-life insurance can be used as an example.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a continuous *df* $F(x)$ and the *pdf* $f(x)$. Let k be a positive integer.

Then the random variables $U^{(k)}(r)$ is given by [Kamps (1995)]

$$U^{(k)}(1) = 1$$
$$U^{(k)}(n+1) = \min\{j > U^{(k)}(n); X_{j,j+k-1} > X_{U^{(k)}(n),U^{(k)}(n)+k-1}\},\ n \in N$$

are called $k-th$ record times and the quantities $X_{U^{(k)}(n)} = X_{U^{(k)}(n),U^{(k)}(n)+k-1}$, $n \in N$ are called $k-th$ record values or k - records.

We can obtain ordinary record values at $k = 1$.

The joint density of the $k-$records $X_{U^{(k)}(1)},\ldots,X_{U^{(k)}(r)}$ is given as

$$f_{X_{U^{(k)}(1)},\ldots,X_{U^{(k)}(r)}}(x_1,\ldots,x_r) = k^r\left(\prod_{i=1}^{r-1}\frac{f(x_i)}{1-F(x_i)}\right)[1-F(x_r)]^{k-1} f(x_r),$$
$$x_1 < \cdots < x_r. \qquad (4.10)$$

and the marginal densities and marginal distribution functions are given by

$$f_{X_{U^{(k)}(r)}}(x) = \frac{k^r}{(r-1)!}[-\ln \overline{F}(x)]^{r-1}[1-F(x)]^{k-1} f(x) \qquad (4.11)$$

and

$$F_{X_{U^{(k)}(r)}}(x) = 1-[1-F(x)]^k \sum_{j=0}^{r-1}\frac{1}{j!}[k\{-\ln \overline{F}(x)\}]^j. \qquad (4.12)$$

5. GENERALIZED ORDER STATISTICS

The model of generalized order statistics (*gos*) have been introduced and extensively studied by Kamps (1995). The use of such model has been steadily growing along the years. This is due to the fact that such model includes several important well known models of ascending ordered random variables that have been separately treated in statistical literature. Order statistics, upper record values, sequential order statistics, progressively Type-II censored order statistics can be discussed as special case of generalized order statistics.

For the developments about the theory and applications of generalized order statistics one may also refer to Kamps and Cramer (2001), Cramer and Kamps (2003), Athar and Islam (2004), Barakat (2007), Barakat *et al.* (2011), Barakat and El-Adll (2012) and references therein.

5.1 Definition and distribution

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x),\ x \in (\alpha, \beta)$.

Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$ and $M_r = \sum_{j=r}^{n-1} m_j$ be the parameters such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1,2,\ldots,n-1\}$. Then $X(r,n,\tilde{m},k),\ r = 1,2,\ldots,n$ are called *gos* if their joint *pdf* is given by [Kamps (1995)]

$$f_{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n)$$

$$= k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{5.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^n$.

In view of (5.1), when $m_i = m;\ i = 1,2,\ldots,n-1$, the *pdf* of $r-th$ *gos* $X(r,n,m,k)$ is given by [Kamps (1995)]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \quad \alpha \le x \le \beta \tag{5.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\overline{F}(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\overline{F}(y)]^{\gamma_s - 1} f(y),$$

$$\alpha \le x < y \le \beta \tag{5.3}$$

where $\overline{F}(x) = 1 - F(x)$

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\log(1-x), & m = -1 \end{cases}$$

$$g_m(x) = \int_0^x (1-t)^m \, dt = h_m(x) - h_m(0), \; x \in [0,1).$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$ is given by

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \frac{[\overline{F}(y)]^{\gamma_s - 1}}{[\overline{F}(x)]^{\gamma_{r+1}}}$$

$$\times \left[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}\right]^{s-r-1} f(y),$$

$$\alpha \le x < y \le \beta \tag{5.4}$$

and the conditional *pdf* of $X(r,n,m,k)$ given $X(s,n,m,k) = y$, $1 \le r < s \le n$ is

$$f_{X(r,n,m,k)|X(s,n,m,k)}(x \mid y) = \frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!} \frac{[\overline{F}(x)]^m \left[1 - (\overline{F}(x))^{m+1}\right]^{r-1}}{\left[1 - (\overline{F}(y))^{m+1}\right]^{s-1}}$$

$$\times \left[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}\right]^{s-r-1} f(x),$$

$$\alpha \le x < y \le \beta. \tag{5.5}$$

Several models of ordered random variables such as order statistics and record values can be seen as special cases of generalized order statistics. If $m=0$ and $k=1$, then $X(r,n,m,k)$ reduces to the $r-th$ order statistics $X_{r:n}$ [David and Nagaraja (2003)]. If $m=-1$ and $k=1$, then $X(r,n,m,k)$ is the $r-th$ record value from an infinite sequence of independent and identically distributed (*iid*) random variables (*rv*) [Ahsanullah (1995)]. Some other special cases are $k-th$ record values ($m=-1$ and k is any positive integer) [Dziubdziela and Kopociński (1976)] and order statistics with non-integral sample size $(m=0,\ k=\alpha-n+1$ and $\alpha\in\mathbb{R}_+$ such that $\alpha>n-1)$ [Stigler (1977), Rohatgi and Saleh (1988)].

5.2 Some important results

Result 1 (Athar and Islam, 2004):

Let $\xi(x)$ is a measurable function of x which is differentiable, then for any arbitrary *df* $F(x)$ and $2\le r\le n$, $n\ge 2$ and $k=1,2,\ldots,$ following relations hold:

When $m_i=m;\ i=1,2,\ldots,n-1$

$$E[\xi\{X(r,n,m,k)\}]-E[\xi\{X(r-1,n,m,k)\}]$$

$$=\frac{C_{r-2}}{(r-1)!}\int_\alpha^\beta \xi'(x)[1-F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))\,dx. \tag{5.6}$$

$$E[\xi\{X(r-1,n,m,k)\}]-E[\xi\{X(r-1,n-1,m,k)\}]$$

$$=-\frac{(m+1)}{\gamma_1}\frac{C_{r-2}^{(n)}}{(r-2)!}\int_\alpha^\beta \xi'(x)[1-F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))dx. \tag{5.7}$$

$$E[\xi\{X(r,n,m,k)\}]-E[\xi\{X(r-1,n-1,m,k)\}]$$

$$=\frac{C_{r-2}^{(n-1)}}{(r-1)!}\int_\alpha^\beta \xi'(x)[1-F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))dx. \tag{5.8}$$

Result 2 (Athar and Islam, 2004):

For $1\le r<s\le n-1,\ n\ge 2$ and $k=1,2,\ldots$

When $m_i=m_j=m;\ i\ne j=1,2,\ldots,n-1$

$$E[\xi\{X(r,n,m,k).X(s,n,m,k)\}]-E[\xi\{X(r,n,m,k).X(s-1,n,m,k)\}]$$

$$=\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)[1-F(x)]^{m}f(x)g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s}\,dydx \tag{5.9}$$

where $\xi(x,y)=\xi_1(x)\xi_2(y)$.

6. DUAL GENERALIZED ORDER STATISTICS

Although the generalized order statistics (*gos*) contain many useful models of ascending ordered random variables. However, the random variables that are in descending order cannot be integrated into this framework. Pawlas and Szynal (2001a) introduced the concept of lower generalized order statistics (*lgos*) to enable a common approach to descending ordered random variables like reverse order statistics and lower record values. Further, Burkschat *et al.* (2003) extensively studied this concept with the name dual generalized order statistics (*dgos*).

For more detailed survey on *dgos* one may refer to Ahsanullah (2004), Mbah and Ahsanullah (2007), Athar *et al.* (2008), Barakat and El-Adll (2009), Khan *et al.* (2010a), Khan *et al.* (2012) and references therein.

6.1 Definition and distribution

Let $X_d(r,n,\tilde{m},k)$, $r=1,2,\ldots,n$ be the $r-th$ *dgos* and their joint *pdf* is given by

$$f_{X_d(1,n,\tilde{m},k),\ldots,X_d(n,n,\tilde{m},k)}(x_1,x_2,\ldots,x_n)$$

$$=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i}f(x_i)\right)[F(x_n)]^{k-1}f(x_n) \tag{6.1}$$

for $F^{-1}(1)>x_1\geq x_2\geq\cdots\geq x_n>F^{-1}(0)$.

If $m_i=0;\ i=1,2,\ldots,n-1$ and $k=1$, we obtain the joint *pdf* of the reverse order statistics and for $m_i=-1,\ k\in N$, we get the joint *pdf* of $k-th$ lower record values.

Here it may be noted that the joint density (6.1) is obtained by replacing $1-F(x)$ with $F(x)$ in (5.1).

In view of (6.1), when $m_i = m;\ i = 1,2,\ldots,n-1$, the *pdf* of $r-th$ *dgos* $X_d(r,n,m,k)$ is given by

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m'^{\,r-1}(F(x)) \tag{6.2}$$

and the joint *pdf* of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{X_d(r,n,m,k),X_d(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)\, g_m'^{\,r-1}(F(x))$$

$$\times [h_m'(F(y)) - h_m'(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y),$$

$$\alpha \le y < x \le \beta, \tag{6.3}$$

where,

$$h_m'(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1}, & m \ne -1 \\ -\log x \quad , & m = -1 \end{cases}$$

and

$$g_m'(x) = h_m'(x) - h_m'(1),\ \ x \in [0,1).$$

The conditional *pdf* of $X_d(s,n,m,k)$ given $X_d(r,n,m,k) = x$, $1 \le r < s \le n$ is given by

$$f_{X_d(s,n,m,k)|X_d(r,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \frac{[F(y)]^{\gamma_s - 1}}{[F(x)]^{\gamma_{r+1}}}$$

$$\times \left[(F(x))^{m+1} - (F(y))^{m+1}\right]^{s-r-1} f(y),$$

$$\alpha \le y < x \le \beta \tag{6.4}$$

and the conditional *pdf* of $X_d(r,n,m,k)$ given $X_d(s,n,m,k) = y$, $1 \le r < s \le n$ is

$$f_{X_d(r,n,m,k)|X_d(s,n,m,k)}(x \mid y) = \frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!} \frac{[F(x)]^m \left[1-(F(x))^{m+1}\right]^{r-1}}{\left[1-(F(y))^{m+1}\right]^{s-1}}$$

$$\times \left[(F(x))^{m+1} - (F(y))^{m+1}\right]^{s-r-1} f(x). \tag{6.5}$$

7. SOME CONTINUOUS DISTRIBUTIONS

7.1 Pareto distribution

A random variable X is said to have the Pareto distribution if its probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$ are of the form given below

$$f(x) = p\lambda^p x^{-(p+1)}; \quad \lambda \le x < \infty; \quad \lambda, p > 0.$$

$$F(x) = 1 - \lambda^p x^{-p}; \quad \lambda \le x < \infty; \quad \lambda, p > 0.$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

7.2 Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below

$$f(x) = p\lambda^{-p} x^{p-1}; \quad 0 \le x < \lambda; \quad \lambda, p > 0.$$

$$F(x) = \lambda^{-p} x^p; \quad 0 \le x < \lambda; \quad \lambda, p > 0.$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = \dfrac{1}{X}$ has a Pareto distribution.

7.3 Beta distribution

i) Beta distribution of first kind

A random variable X is said to have the beta distribution of first kind if its *pdf* is of the form

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}; \quad 0 \le x \le 1, \; p, q > 0.$$

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q=1$, the distribution reduces to power function distribution.

ii) Beta distribution of second kind

The continuous random variable X which is distributed according to probability law:

$$f(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}}; \quad 0 \le x < \infty; \quad p,q > 0.$$

is known as a beta variate of the second kind with parameters p and q.

Remark: Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $\frac{1}{y}$.

Usage: The Beta distribution is one of the most frequently employed distribution to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

7.4 Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta e^{-\theta x}; \quad 0 \le x < \infty; \quad \theta > 0$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x}; \quad 0 \le x < \infty; \quad \theta > 0.$$

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s+t | X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

Then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

7.5 Weibull distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by

$$f(x) = \theta\, p\, x^{p-1} e^{-\theta x^p}; \quad 0 \le x < \infty; \ \theta, p > 0.$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^p}; \quad 0 \le x < \infty; \ \theta, p > 0.$$

Remark 7.5.1: If we put $p = 1$ in the *pdf* of Weibull distribution, we get the *pdf* of exponential distribution.

Remark 7.5.2: If we put $p = 2$, it gives the *pdf* of Rayleigh distribution.

Remark 7.5.3: If X has a Weibull distribution, then the *pdf* of

$Y = -\log(\theta x^p)$ is

$$f(y) = e^{-y} e^{-e^{-y}}$$

which is a form of an Extreme Value distribution.

Remark 7.5.4: The *pdf* and the *df* of inverse Weibull distribution is given by

$$f(x) = \theta\, p\, x^{-(p+1)}\, e^{-\theta x^{-p}}; \quad 0 \le x < \infty; \;\; \theta, p > 0.$$

$$F(x) = e^{-\theta\, x^{-p}}; \quad 0 \le x < \infty; \;\; \theta, p > 0.$$

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

7.6 Rectangular distribution

A random variable X is said to have a rectangular distribution if its *pdf* is given by

$$f(x) = \frac{1}{\lambda - \beta}; \;\; \beta \le x \le \lambda$$

and the *df* is given by

$$F(x) = \frac{x - \beta}{\lambda - \beta}; \;\; \beta \le x \le \lambda.$$

The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(x)$ follows rectangular distribution $R(0,1)$. This distribution is used in “rounding off” errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

7.7 Burr distribution

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below [Johnson *et al.* (1994)]

i) $F(x) = x, \;\; 0 < x < 1$

ii) $F(x) = (1 + e^{-x})^{-\gamma}, \quad -\infty < x < \infty$

iii) $F(x) = (1 + x^{-c})^{-\gamma}, \quad 0 \le x < \infty$

iv) $F(x) = \left[1 + \left(\dfrac{c - x}{x}\right)^{1/c}\right]^{-\gamma}, \quad 0 < x < c$

v) $F(x) = [1 + ce^{-\tan x}]^{-\gamma}, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

vi) $F(x) = [1 + ce^{-\gamma \sinh x}]^{-\gamma}, \quad -\infty < x < \infty$

vii) $F(x) = 2^{-\gamma}(1 + \tanh x)^{\gamma}, \quad -\infty < x < \infty$

viii) $F(x) = \left(\dfrac{2}{\pi}\tan^{-1} e^{x}\right)^{\gamma}, \quad -\infty < x < \infty$

ix) $F(x) = 1 - \left[\dfrac{2}{c[(1 + e^{x})^{\gamma} - 1] + 2}\right], \quad -\infty < x < \infty$

x) $F(x) = (1 - e^{-x^2})^{\gamma}, \quad 0 \le x < \infty$

xi) $F(x) = \left(x - \dfrac{1}{2\pi}\sin 2\pi x\right)^{\gamma}, \quad 0 < x < 1$

xii) $F(x) = 1 - (1 + x^{c})^{-\gamma}, \quad 0 < x < \infty,$

where γ and c are positive parameters.

Special attention is given to type XII, whose *pdf* is given as:

$$f(x) = \gamma c x^{c-1}(1 + x^{c})^{-(\gamma+1)}; \;\; 0 < x < \infty; \;\; c, \gamma > 0.$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c = 1$, it is called Lomax distribution whereas at $\gamma = 1$, it is known as Log-logistic distribution.

7.8 Cauchy distribution

The special form of the Pearson type VII distribution, with the *pdf*

$$f(x) = \frac{1}{\pi\lambda}\frac{1}{[1 + \{(x - \theta)/\lambda\}^2]}; \; -\infty < x < \infty; \; \lambda > 0; -\infty < \theta < \infty$$

is called the Cauchy distribution.

The *df* is given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right); \; -\infty < x < \infty; \; \lambda > 0; \; -\infty < \theta < \infty .$$

The distribution is symmetrical about $x = \theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta = 0$, $\lambda = 1$. The standard probability density function is given by

$$f(x) = \frac{1}{\pi}\frac{1}{(1+x^2)}; \; -\infty < x < \infty$$

and the standard cumulative distribution function is

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x; \; -\infty < x < \infty .$$

CHAPTER II

*MOMENT PROPERTIES OF PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS FROM DOUBLY TRUNCATED WEIBULL DISTRIBUTION

1. INTRODUCTION

The progressive censoring and associated inferential procedures have been extensively studied in the literature for a number of distributions by several authors, such as Cohen (1963, 1966, 1975, 1976 and 1991), Mann (1969, 1971), Cohen and Whitten (1982, 1988), Viveros and Balakrishnan (1994), Balakrishnan and Sandhu (1995), Ali Mousa and Jaheen (2002), Kim and Han (2010) and Kim *et al.* (2011). Aggarwala and Balakrishnan (1996) and Balakrishnan and Aggarwala (2000) have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distribution and their truncated forms. Also, Saran and Pushkarna (2001) have obtained several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from the doubly truncated Burr distribution. Mahmoud *et al.* (2006) derived some new recurrence relations for single and product moments of progressively Type-II right censored order statistics from the linear exponential distribution and also obtained maximum likelihood estimators (MLEs) of the location and scale parameters. Balakrishnan *et al.* (2011) and Balakrishnan and Saleh (2011, 2012 and 2013) have established several recurrence relations for single and product moments of progressively Type-II right censored order statistics from logistic, half-logistic, log-logistic and generalized half logistic distributions and the moments so determined are then utilized in inferential method to derive best linear unbiased estimators of the scale and location parameters.

For the developments about the theory and applications of progressive censoring, one may refer to Balakrishnan and Aggarwala (2000), Balakrishnan (2007), Balakrishnan and Cramer (2014) and references therein.

*Part of the results of this chapter are contained in Athar and Akhter (2015a)

The Weibull distribution is named after the Swedish physicist, Weibull (1939a, 1939b), who used it to represent the distribution of the breaking strength of materials. The Weibull distribution is one of the most widely used distributions for analyzing lifetime data. It is to be found useful in diverse fields ranging from engineering to medical sciences [see Martz and Waller (1982), Lawless (2003)]. For a wide variety of other applications of the Weibull distribution one may refer to Johnson *et al.* (1994, 1995) and Rinne (2009).

Recurrence relations for moments of order statistics for Weibull distribution as well as doubly truncated Weibull distribution are investigated by Lieblein (1955), Balakrishnan and Joshi (1981a), Khan *et al.* (1983a, 1983b), Khan *et al.* (1984), Ali and Khan (1996) and Balakrishnan and Sultan (1998), among others.

In this chapter, recurrence relations for the single and product moments of progressively Type-II right censored order statistics arising from the doubly truncated Weibull distribution are obtained. These relations generalize the results given by Balakrishnan *et al.* (2001) for the progressively Type-II right censored order statistics for standard exponential and right truncated exponential distributions. These results, also generalize the corresponding results for usual order statistics due to Khan *et al.* (1983a, 1983b) for the doubly truncated Weibull distribution. These relations, when used in a systematic manner, enable the recursive computation of moments for all sample sizes and all possible progressive censoring schemes.

Let $X_{1:m:n}^{\tilde{R}}, X_{2:m:n}^{\tilde{R}}, \ldots, X_{m:m:n}^{\tilde{R}}$ be the m ordered observed failure times in a sample of size n under progressive Type-II right censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$ from the doubly truncated Weibull distribution with *pdf*

$$f(x) = \frac{p\,x^{p-1} e^{-x^p}}{P-Q}, \quad -\log(1-Q) \le x^p \le -\log(1-P), \quad p > 0, \tag{1.1}$$

The exponential and Rayleigh distributions are considered as special cases of the Weibull distribution when $p = 1$ and $p = 2$ respectively.

The *df* corresponding to (1.1) is,

$$F(x) = \frac{1}{(P-Q)}\left(e^{-Q_1{}^p} - e^{-x^p}\right), \tag{1.2}$$

where $1-P$ and Q are the proportion of truncation on the right and the left of the distribution, respectively.

Here, $Q_1^p = -\log(1-Q)$ and $P_1^p = -\log(1-P)$

Let

$$P_2 = \frac{(1-P)}{(P-Q)} \text{ and } Q_2 = \frac{(1-Q)}{(P-Q)}.$$

then, it can be seen that from (1.1) and (1.2),

$$f(x) = px^{p-1}\{P_2 + [1-F(x)]\}. \tag{1.3}$$

The relation in (1.3) is the "characterizing differential equation" for the distribution in (1.1).

Now, we shall denote

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}} = E[X_{r:m:n}^{\tilde{R}}]^i = \mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(i)}},$$

$$1 \le r \le m \le n,\ i \ge 0,\ R_r^* - 1 \ge m-r \ge 0 \tag{1.4}$$

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = E[\{X_{r:m:n}^{\tilde{R}}\}^i\{X_{s:m:n}^{\tilde{R}}\}^j] = \mu_{r,s:s:n}^{(R_1,\ldots,R_{s-1},R_s^*-1)^{(i,j)}}$$

$$1 \le r < s \le m \le n,\ i, j \ge 0, \tag{1.5}$$

as given by Balakrishnan *et al.* (2001).

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

In this section, we shall exploit the relation (1.3) to derive recurrence relations for the single moments of progressively Type-II right censored order statistics from the doubly truncated Weibull distribution. In view of (1.3.1) the $i-th$ single moment is given as

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}} = E[X_{r:m:n}^{\tilde{R}}]^i$$

$$= c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m \le P_1} x_r^i f(x_1)[1-F(x_1)]^{R_1}$$

$$\times f(x_2)[1-F(x_2)]^{R_2} \cdots f(x_m)[1-F(x_m)]^{R_m} dx_1 \cdots dx_m, \quad (2.1)$$

where $c(n,m-1)$ is as defined in (1.3.2). The single moments of progressively Type-II right censored order statistics in (2.1) satisfy the following recurrence relations.

Theorem 2.1: *For the doubly truncated Weibull distribution as given in* (1.1) *and for* $2 \le m \le n-1$ *and* $i \ge 0$,

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i+p)}} = \mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(i+p)}} + \frac{1}{R_m^*}\Bigg[\frac{(i+p)}{p}\mu_{m:m:n}^{(R_1,\ldots,R_{m-1},R_m^*-1)^{(i)}}$$

$$+P_2\Bigg\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(i+p)}}$$

$$-\frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(i+p)}}\Bigg\}\Bigg]. \quad (2.2)$$

Proof: From (2.1), we have

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i)}} = c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_{m-1} < P_1} Z(x_{m-1}) f(x_1)[1-F(x_1)]^{R_1}$$

$$\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1 \cdots dx_{m-1}, \quad (2.3)$$

where

$$Z(x_{m-1}) = \int_{x_{m-1}}^{P_1} x_m^i [1-F(x_m)]^{R_m} f(x_m) dx_m. \quad (2.4)$$

Now using relation (1.3) in (2.4), we get

$$Z(x_{m-1}) = p\Bigg[P_2\int_{x_{m-1}}^{P_1} x_m^{i+p-1}[1-F(x_m)]^{R_m} dx_m$$

$$+\int_{x_{m-1}}^{P_1} x_m^{i+p-1}[1-F(x_m)]^{R_m+1} dx_m\Bigg] \quad (2.5)$$

$$Z(x_{m-1}) = \frac{p}{(i+p)}\left[P_2 \left\{ -x_{m-1}^{i+p}[1-F(x_{m-1})]^{R_m} \right.\right.$$

$$\left. + R_m \int_{x_{m-1}}^{P_1} x_m^{i+p}[1-F(x_m)]^{R_m-1} f(x_m)\, dx_m \right\}$$

$$- x_{m-1}^{i+p}[1-F(x_{m-1})]^{R_m+1}$$

$$\left. + (R_m+1)\int_{x_{m-1}}^{P_1} x_m^{i+p}[1-F(x_m)]^{R_m} f(x_m)\, dx_m \right]. \tag{2.6}$$

Substituting the resulting expression of $Z(x_{m-1})$ from (2.6) in (2.3), we get

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i)}} = \frac{p}{(i+p)}\left[P_2 \left\{ -c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_{m-1} < P_1} x_{m-1}^{i+p} f(x_1)[1-F(x_1)]^{R_1} \right.\right.$$

$$\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m} dx_1\cdots dx_{m-1}$$

$$+ R_m\, c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m < P_1} x_m^{i+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\left. \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} f(x_m)[1-F(x_m)]^{R_m-1} dx_1\cdots dx_m \right\}$$

$$- c(n,m-1)\iint\ldots\int_{Q_1 \le x_1 < \cdots < x_{m-1} < P_1} x_{m-1}^{i+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m+1} dx_1\cdots dx_{m-1}$$

$$+ (R_m+1)\; c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m < P_1} x_m^{i+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\left. \times\cdots\times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} f(x_m)[1-F(x_m)]^{R_m} dx_1\cdots dx_m \right]$$

$$= \frac{p}{(i+p)}\left[P_2 \left\{ -\frac{c(n,m-1)}{c(n-1,m-2)} \mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(i+p)}} \right.\right.$$

$$\left. + \frac{c(n,m-1)}{c(n-1,m-1)} R_m\, \mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(i+p)}} \right\}$$

$$- (n - R_1 - \cdots - R_{m-1} - m + 1)\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i+p)}}$$

$$\left. + (R_m+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i+p)}} \right].$$

After rearranging the terms, we get

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i+p)}}$$

$$= \frac{1}{(R_m+1)}\left[\frac{(i+p)}{p}\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i)}} + P_2\left\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(i+p)}}\right.\right.$$

$$\left.\left. - \frac{c(n,m-1)}{c(n-1,m-1)}R_m\,\mu_{m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(i+p)}}\right\}\right] + \mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i+p)}}, \quad (2.7)$$

and, hence the theorem.

Theorem 2.2: *For* $2 \le r \le m-1,\ m \le n-1$ *and* $i \ge 0$,

$$\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(i+p)}} = \mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(i+p)}}$$

$$= \mu_{r-1:r-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-1)^{(i+p)}} + \frac{1}{R_r^*}\left[\frac{(i+p)}{p}\mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(i)}}\right.$$

$$+ P_2\left\{\frac{c(n,r-1)}{c(n-1,r-2)}\mu_{r-1:r-1:n-1}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-2)^{(i+p)}}\right.$$

$$\left.\left. - \frac{c(n,r-1)}{c(n-1,r-1)}(R_r^*-1)\mu_{r:r:n-1}^{(R_1,\ldots,R_{r-1},R_r^*-2)^{(i+p)}}\right\}\right]. \quad (2.8)$$

Proof: This follows from Theorem 2.1 and (1.4).

Similarly, it can be seen that

$$\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(i+p)}} = \mu_{1:1:n}^{(R_1^*-1)^{(i+p)}}$$

$$= \frac{(i+p)}{np}\mu_{1:1:n}^{(R_1^*-1)^{(i)}} - P_2\mu_{1:1:n-1}^{(R_1^*-2)^{(i+p)}} + Q_2Q_1^{i+p}, \quad n \ge 2 \quad (2.9)$$

and

$$\mu_{1:1:1}^{(0)^{(i+p)}} = \frac{(i+p)}{p}\mu_{1:1:1}^{(0)^{(i)}} - P_2P_1^{i+p} + Q_2Q_1^{i+p} = \mu_{1:1}^{(i+p)}, \quad n = 1. \quad (2.10)$$

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

For any continuous distribution, the $i-th$ and the $j-th$ product moment of the progressively Type-II right censored order statistics, in view of (1.3.1), is given as

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = E[\{X_{r:m:n}^{\tilde{R}}\}^i \{X_{s:m:n}^{\tilde{R}}\}^j]$$

$$= c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m \le P_1} x_r^i\, x_s^j\, f(x_1)[1-F(x_1)]^{R_1}$$

$$\times f(x_2)[1-F(x_2)]^{R_2} \cdots f(x_m)[1-F(x_m)]^{R_m}\, dx_1 \cdots dx_m\,. \quad (3.1)$$

We shall again exploit relation (1.3) to obtain the recurrence relations for product moments of progressively Type-II right censored order statistics from the doubly truncated Weibull distribution.

Theorem 3.1: *For* $1 \le r < m \le n-1$ *and* $i, j \ge 0$,

$$\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j+p)}} = \mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-1)^{(i,j+p)}} + \frac{1}{R_m^*}\left[\frac{(j+p)}{p}\mu_{r,m:m:n}^{(R_1,\ldots,R_{m-1},R_m^*-1)^{(i,j)}}\right.$$

$$+ P_2\left\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{r,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}^*-2)^{(i,j+p)}}\right.$$

$$\left.\left. - \frac{c(n,m-1)}{c(n-1,m-1)}(R_m^*-1)\mu_{r,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m^*-2)^{(i,j+p)}}\right\}\right]. \quad (3.2)$$

Proof: From (3.1), we have

$$\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j)}} = c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_{m-1} \le P_1} x_r^i Z(x_{m-1}) f(x_1)[1-F(x_1)]^{R_1}$$

$$\times \cdots \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}}\, dx_1 \cdots dx_{m-1}\,, \quad (3.3)$$

where $Z(x_{m-1})$ is same as in (2.4).

Now substituting the value of $Z(x_{m-1})$ from (2.6) after replacing i with j, we get

$$\mu_{r,m:m:n}^{(R_1,\dots,R_m)^{(i,j)}}$$

$$= \frac{p}{(j+p)}\left[P_2 \left\{ -c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_{m-1} < P_1} x_r^i \, x_{m-1}^{j+p} f(x_1)[1-F(x_1)]^{R_1} \right.\right.$$

$$\times \cdots \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m} dx_1 \cdots dx_{m-1}$$

$$+ R_m \, c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m < P_1} x_r^i \, x_m^{j+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\left. \times \cdots \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} f(x_m)[1-F(x_m)]^{R_m-1} dx_1 \cdots dx_m \right\}$$

$$- c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_{m-1} < P_1} x_r^i \, x_{m-1}^{j+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\times \cdots \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}+R_m+1} dx_1 \cdots dx_{m-1}$$

$$+ (R_m+1)\, c(n,m-1)\iint\cdots\int_{Q_1 \le x_1 < \cdots < x_m < P_1} x_r^i \, x_m^{j+p} f(x_1)[1-F(x_1)]^{R_1}$$

$$\left. \times \cdots \times f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} f(x_m)[1-F(x_m)]^{R_m} dx_1 \cdots dx_m \right],$$

$$= \frac{p}{(j+p)}\left[P_2 \left\{ -\frac{c(n,m-1)}{c(n-1,m-2)} \mu_{r,m-1:m-1:n-1}^{(R_1,\dots,R_{m-2},R_{m-1}+R_m)^{(i,j+p)}} \right.\right.$$

$$\left. + \frac{c(n,m-1)}{c(n-1,m-1)} R_m \mu_{r,m:m:n-1}^{(R_1,\dots,R_{m-1},R_m-1)^{(i,j+p)}} \right\}$$

$$- (n - R_1 - \cdots - R_{m-1} - m + 1)\mu_{r,m-1:m-1:n}^{(R_1,\dots,R_{m-2},R_{m-1}+R_m+1)^{(i,j+p)}}$$

$$\left. + (R_m+1)\mu_{m:m:n}^{(R_1,\dots,R_m)^{(i,j+p)}} \right].$$

After rearranging the terms, we get

$$\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j+p)}}$$

$$=\frac{1}{(R_m+1)}\left[\frac{(j+p)}{p}\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}+P_2\left\{\frac{c(n,m-1)}{c(n-1,m-2)}\mu_{r,m-1:m-1:n-1}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m)^{(i,j+p)}}\right.\right.$$

$$\left.\left.-\frac{c(n,m-1)}{c(n-1,m-1)}R_m\,\mu_{r,m:m:n-1}^{(R_1,\ldots,R_{m-1},R_m-1)^{(i,j+p)}}\right\}\right]+\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i,j+p)}}. \quad (3.4)$$

Hence the (3.2).

Theorem 3.2: *For* $1\le r<s\le n-1$ *and* $i,j\ge 0$,

$$\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i,j+p)}}=\mu_{r,s:s:n}^{(R_1,\ldots,R_{s-1},R_s^*-1)^{(i,j+p)}}$$

$$=\mu_{r,s-1:s-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}^*-1)^{(i,j+p)}}+\frac{1}{R_s^*}\left[\frac{(j+p)}{p}\mu_{r,s:s:n}^{(R_1,\ldots,R_{s-1},R_s^*-1)^{(i,j)}}\right.$$

$$+P_2\left\{\frac{c(n,s-1)}{c(n-1,s-2)}\mu_{r,s-1:s-1:n-1}^{(R_1,\ldots,R_{s-2},R_{s-1}^*-2)^{(i,j+p)}}\right.$$

$$\left.\left.-\frac{c(n,s-1)}{c(n-1,s-1)}(R_s^*-1)\mu_{r,s:s:n-1}^{(R_1,\ldots,R_{s-1},R_s^*-2)^{(i,j+p)}}\right\}\right]. \quad (3.5)$$

Proof: This follows from Theorem 3.1 and (1.5).

4. REMARKS

Remark 4.1: By letting $p=1$, we can get the corresponding recurrence relations for the doubly truncated exponential distribution and at $p=2$, we can get the corresponding recurrence relations for the doubly truncated Rayleigh distribution.

Remark 4.2: By letting $Q\to 0$ and $P\to 1$ in the Theorems, we can get the corresponding recurrence relations for the non-truncated distribution as

$$\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(i+p)}}=\mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(i+p)}}$$

$$=\frac{1}{R_r^*}\frac{(i+p)}{p}\mu_{r:r:n}^{(R_1,\ldots,R_{r-1},R_r^*-1)^{(i)}}+\mu_{r-1:r-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}^*-1)^{(i+p)}}$$

and

$$\mu_{r,s:m:n}^{(R_1,\dots,R_m)^{(i,j+p)}} = \mu_{r,s:s:n}^{(R_1,\dots,R_{s-1},R_s^*-1)^{(i,j+p)}}$$

$$= \frac{1}{R_s^*}\frac{(j+p)}{p}\mu_{r,s:s:n}^{(R_1,\dots,R_{s-1},R_s^*-1)^{(i,j)}} + \mu_{r,s-1:s-1:n}^{(R_1,\dots,R_{s-2},R_{s-1}^*-1)^{(i,j+p)}} .$$

Remark 4.3: For $R_1 = R_2 = \cdots = R_m = 0$, so that $m = n$, the results established would reduce to the order statistics,

$$\mu_{r:n}^{(i+p)} = \frac{(i+p)}{np}\mu_{r:n}^{(i)} - P_2\mu_{r:n-1}^{(i+p)} + Q_2\mu_{r-1:n-1}^{(i+p)}, \quad 2 \le r \le n-1.$$

and

$$\mu_{r,s:n}^{(i,j+p)} = \mu_{r,s-1:n}^{(i,j+p)} + \frac{(j+p)}{p(n-s+1)}\mu_{r,s:n}^{(i,j)} - \frac{nP_2}{(n-s+1)}\left\{\mu_{r,s:n-1}^{(i,j+p)} - \mu_{r,s-1:n-1}^{(i,j+p)}\right\},$$

$$1 \le r < s \le n, \;\; s-r \ge 2 .$$

as given by Khan *et al.* (1983a, 1983b) and Balakrishnan and Sultan (1998).

Remark 4.4: At $i = 0$ in Theorem 3.2, we get the results as established in Theorem 2.2.

CHAPTER III

*MOMENTS OF PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS FROM LINDLEY DISTRIBUTION

1. INTRODUCTION

The exponential distribution is a basic physical model in reliability theory and survival analysis due to the popularity of the exponential distribution in statistics and many applied areas. Lindley distribution was first introduced by Lindley (1958) in connection with the Fiducial distribution and Bayes theorem. The Lindley distribution is important for studying stress-strength reliability modeling [Hussain (2006)]. Ghitany *et al.* (2008) was the first who studied the statistical properties of the said distribution and have shown that in many ways the Lindley distribution is a better model than that based on the exponential distribution.

In this chapter, some recurrence relations between moments of progressively Type-II right censored order statistics from Lindley distribution are obtained. Further, computational algorithm to compute moments is also given.

Let $X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ be the m ordered observed failure times in a sample of size n under progressive Type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$ from the Lindley distribution with the *pdf* ,

$$f(x) = \frac{\lambda^2}{1+\lambda}(1+x)e^{-\lambda x}, \; x>0, \; \lambda>0, \tag{1.2}$$

and the df

$$F(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x}, \; x>0, \; \lambda>0. \tag{1.3}$$

It may be noted that from (1.2) and (1.3),

$$(1+\lambda+\lambda x) f(x) = \lambda^2 \left\{[1-F(x)] + x[1-F(x)]\right\}. \tag{1.4}$$

The relation in (1.4) is the "characterizing differential equation" for the distribution in (1.2).

*Part of the results of this chapter are contained in Athar *et al.* (2014)

2. MAIN RESULTS

For any continuous distribution, we shall denote the $i-th$ single moment of the progressively Type-II right censored order statistics in view of (1.3.1) as

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}} = E[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}]^i$$
$$= c(n,m-1)\iint\ldots\int_{0<x_1<\cdots<x_m<\infty} x_r^i f(x_1)[1-F(x_1)]^{R_1}$$
$$\times f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m)[1-F(x_m)]^{R_m} dx_1\ldots dx_m, \quad (2.1)$$

and the $i-th$ and the $j-th$ product moments as

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}} = E[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}} X_{s:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}}]$$
$$= c(n,m-1)\iint\ldots\int_{0<x_1<\cdots<x_m<\infty} x_r^i x_s^j f(x_1)[1-F(x_1)]^{R_1}$$
$$\times f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m)[1-F(x_m)]^{R_m} dx_1\ldots dx_m \quad (2.2)$$

where $c(n,m-1)$ is as defined in (1.3.2).

Theorem 2.1: *For* $s\geq 2,\ 3\leq m\leq n$ *and* $i,\ j\geq 0,$

$$\mu_{1,s:m:n}^{(R_1,\ldots,R_m)^{(i+2,j)}} = \frac{(i+2)}{\lambda(R_1+1)}\left[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{1,s:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}\right.$$
$$\left.+\left\{1-\frac{\lambda(R_1+1)}{(i+1)}\right\}\mu_{1,s:m:n}^{(R_1,\ldots,R_m)^{(i+1,j)}}\right]$$
$$-\frac{(n-R_1-1)}{(R_1+1)}\left[\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(i+2,j)}}\right.$$
$$\left.+\left\{\frac{i+2}{i+1}\right\}\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(i+1,j)}}\right]. \quad (2.3)$$

Proof: From (2.2), we have

$$(1+\lambda)\mu_{1,s:m:n}^{(R_1,\ldots,R_m)^{(i,j)}} + \lambda\mu_{1,s:m:n}^{(R_1,\ldots,R_m)^{(i+1,j)}}$$
$$= c(n,m-1)\iint\ldots\int_{0<x_2<\cdots<x_m<\infty} x_s^j Z(x_2) f(x_2)[1-F(x_2)]^{R_2}$$
$$\times\cdots\times f(x_m)[1-F(x_m)]^{R_m} dx_2\ldots dx_m, \quad (2.4)$$

where,

$$Z(x_2)=\int_0^{x_2} x_1^i(1+\lambda+\lambda x_1)\,f(x_1)[1-F(x_1)]^{R_1}\,dx_1. \tag{2.5}$$

On using relation (1.4) in (2.5) and integrating by parts, we get

$$\begin{aligned}Z(x_2)=\lambda^2\Bigg[&\frac{1}{(i+1)}\Bigg\{x_2^{i+1}[1-F(x_2)]^{R_1+1}\\&+(R_1+1)\int_0^{x_2} x_1^{i+1}[1-F(x_1)]^{R_1}\,f(x_1)\,dx_1\Bigg\}\\&+\frac{1}{(i+2)}\Bigg\{x_2^{i+2}[1-F(x_2)]^{R_1+1}\\&+(R_1+1)\int_0^{x_2} x_1^{i+2}[1-F(x_1)]^{R_1}\,f(x_1)\,dx_1\Bigg\}\Bigg].\end{aligned} \tag{2.6}$$

Now by substituting the resultant expression of $Z(x_2)$ from (2.6) in (2.4) and simplifying, we get the required result.

Corollary 2.1: *For* $1\le m\le n$ *and* $i\ge 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(i+2)}}&=\frac{(i+2)}{\lambda(R_1+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(i)}}\\&\quad+\left\{1-\frac{\lambda(R_1+1)}{(i+1)}\right\}\mu_{1:m:n}^{(R_1,\ldots,R_m)^{(i+1)}}\Bigg]\\&\quad-\frac{(n-R_1-1)}{(R_1+1)}\Bigg[\mu_{1:m-1:n}^{(R_1+R_2+1,\,R_3,\ldots,R_m)^{(i+2)}}\\&\quad+\left\{\frac{i+2}{i+1}\right\}\mu_{1:m-1:n}^{(R_1+R_2+1,\,R_3,\ldots,R_m)^{(i+1)}}\Bigg]\end{aligned} \tag{2.7}$$

and subsequently for $m=1$,

$$\mu_{1:1:n}^{(n-1)^{(i+2)}}=\frac{(i+2)}{n\lambda}\left[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{1:1:n}^{(n-1)^{(i)}}+\left\{1-\frac{n\lambda}{(i+1)}\right\}\mu_{1:1:n}^{(n-1)^{(i+1)}}\right]. \tag{2.8}$$

Proof: Relation (2.7) is the simple consequence of Theorem 2.1 and can be established by putting $j=0$ in (2.3). Further, (2.8) can be seen by putting $m=1$ in (2.7) after noting that $R_1=n-1$.

Corollary 2.2: *For* $2 \le m \le n$ *and* $i, j \ge 0$,

$$\mu_{1,2:m:n}^{(R_1,\dots,R_m)^{(i+2,j)}} = \frac{(i+2)}{\lambda(R_1+1)}\Bigg[\frac{(1+\lambda)}{\lambda}\mu_{1,2:m:n}^{(R_1,\dots,R_m)^{(i,j)}} + \left\{1-\frac{\lambda(R_1+1)}{(i+1)}\right\}\mu_{1,2:m:n}^{(R_1,\dots,R_m)^{(i+1,j)}}\Bigg]$$
$$-\frac{(n-R_1-1)}{(R_1+1)}\Bigg[\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(i+j+2)}} + \left\{\frac{i+2}{i+1}\right\}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(i+j+1)}}\Bigg] \qquad (2.9)$$

Proof: (2.9) can be proved by putting $s=2$ in (2.3).

Theorem 2.2: *For* $1 \le r < s \le m-1$, $m \le n$ *and* $i, j \ge 0$,

$$\mu_{r,s:m:n}^{(R_1,\dots,R_m)^{(i,j+2)}} = \frac{(j+2)}{\lambda(R_s+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r,s:m:n}^{(R_1,\dots,R_m)^{(i,j)}} + \left\{1-\frac{\lambda(R_s+1)}{(j+1)}\right\}\mu_{r,s:m:n}^{(R_1,\dots,R_m)^{(i,j+1)}}\Bigg]$$
$$-\frac{(n-R_1-\cdots-R_s-s)}{(R_s+1)}\Bigg[\mu_{r,s:m-1:n}^{(R_1,\dots,R_{s-1},R_s+R_{s+1}+1,\dots,R_m)^{(i,j+2)}} + \left\{\frac{j+2}{j+1}\right\}\mu_{r,s:m-1:n}^{(R_1,\dots,R_{s-1},R_s+R_{s+1}+1,\dots,R_m)^{(i,j+1)}}\Bigg]$$
$$+\frac{(n-R_1-\cdots-R_{s-1}-s+1)}{(R_s+1)}\Bigg[\mu_{r,s-1:m-1:n}^{(R_1,\dots,R_{s-2},R_{s-1}+R_s+1,\dots,R_m)^{(i,j+2)}} + \left\{\frac{j+2}{j+1}\right\}\mu_{r,s-1:m-1:n}^{(R_1,\dots,R_{s-2},R_{s-1}+R_s+1,\dots,R_m)^{(i,j+1)}}\Bigg]. \qquad (2.10)$$

Proof: From (2.2), we get

$$(1+\lambda)\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}+\lambda\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i,j+1)}}$$

$$=(1+\lambda)E\left[X_{r:m:n}^{(R_1,\ldots,R_m)^{(i)}}X_{s:m:n}^{(R_1,\ldots,R_m)^{(j)}}\right]+\lambda E\left[X_{r:m:n}^{(R_1,\ldots,R_m)^{(i)}}X_{s:m:n}^{(R_1,\ldots,R_m)^{(j+1)}}\right]$$

$$=c(n,m-1)\iint\ldots\int_{0<x_1<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}x_r^i Z(x_{s-1},x_{s+1})f(x_1)[1-F(x_1)]^{R_1}$$

$$\times\cdots\times f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}$$

$$\times\cdots\times f(x_m)[1-F(x_m)]^{R_m}dx_1\ldots dx_{s-1}dx_{s+1}\ldots dx_m, \tag{2.11}$$

where

$$Z(x_{s-1},x_{s+1})=\int_{x_{s-1}}^{x_{s+1}}x_s^j(1+\lambda+\lambda x_s)f(x_s)[1-F(x_s)]^{R_s}dx_s. \tag{2.12}$$

On using relation (1.4) in (2.12) and integrating by parts, we get

$$\begin{aligned}Z(x_{s-1},x_{s+1}) &=\lambda^2\Bigg[\frac{1}{(j+1)}\Bigg(x_{s+1}^{j+1}[1-F(x_{s+1})]^{R_s+1}-x_{s-1}^{j+1}[1-F(x_{s-1})]^{R_s+1}\\&\quad+(R_s+1)\int_{x_{s-1}}^{x_{s+1}}x_s^{j+1}[1-F(x_s)]^{R_s}f(x_s)dx_s\Bigg)\\&\quad+\frac{1}{(j+2)}\Bigg(x_{s+1}^{j+2}[1-F(x_{s+1})]^{R_s+1}-x_{s-1}^{j+2}[1-F(x_{s-1})]^{R_s+1}\\&\quad+(R_s+1)\int_{x_{s-1}}^{x_{s+1}}x_s^{j+2}[1-F(x_s)]^{R_s}f(x_s)dx_s\Bigg)\Bigg].\end{aligned} \tag{2.13}$$

Now upon substituting the resultant expression of $Z(x_{s-1},x_{s+1})$ from (2.13) in (2.11) and simplifying, yields (2.10).

Corollary 2.3: *For* $2\le r\le m-1,\ m\le n$ *and* $i\ge 0,$

$$\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(i+2)}}=\frac{(i+2)}{\lambda(R_r+1)}\left[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(i)}}+\left\{1-\frac{\lambda(R_r+1)}{(i+1)}\right\}\mu_{r:m:n}^{(R_1,\ldots,R_m)^{(i+1)}}\right]$$

$$-\frac{(n-R_1-\cdots-R_r-r)}{(R_r+1)}\Bigg[\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(i+2)}}$$

$$+\left\{\frac{i+2}{i+1}\right\}\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(i+1)}}\Bigg]$$

$$+\frac{(n-R_1-\cdots-R_{r-1}-r+1)}{(R_r+1)}\Bigg[\mu_{r-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(i+2)}}$$

$$+\left\{\frac{i+2}{i+1}\right\}\mu_{r-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(i+1)}}\Bigg]. \tag{2.14}$$

Proof: Relation (2.14) can be established by putting $i=0$ in (2.10) and replacing s by r.

Corollary 2.4: *For* $2\le m\le n$ *and* $i\ge 0$,

$$\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i+2)}}=\frac{(i+2)}{\lambda(R_m+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i)}}$$

$$+\left\{1-\frac{\lambda(R_m+1)}{(i+1)}\right\}\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(i+1)}}\Bigg]$$

$$+\frac{(n-R_1-\cdots-R_{m-1}-m+1)}{(R_m+1)}\Bigg[\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i+2)}}$$

$$+\left\{\frac{i+2}{i+1}\right\}\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i+1)}}\Bigg]. \tag{2.15}$$

Proof: (2.15) can be proved by replacing r with m in (2.14) and noting that $\mu_{m:m-1:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}=0$ or since, $n=m+\sum_{t=1}^{m}R_m \Rightarrow n-R_1-\cdots-R_m-m=0.$

Corollary 2.5: *For* $1\le r\le m-2$, $m\le n$ *and* $i, j\ge 0$,

$$\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)^{(i,j+2)}}=\frac{(j+2)}{\lambda(R_{r+1}+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}$$

$$+\left\{1-\frac{\lambda(R_{r+1}+1)}{(j+1)}\right\}\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)^{(i,j+1)}}\Bigg]$$

$$-\frac{(n-R_1-\cdots-R_{r+1}-r-1)}{(R_{r+1}+1)}\Bigg[\mu_{r,r+1:m-1:n}^{(R_1,\ldots,R_r,R_{r+1}+R_{r+2}+1,\ldots,R_m)^{(i,j+2)}}$$

$$+\left\{\frac{j+2}{j+1}\right\}\mu_{r,r+1:m-1:n}^{(R_1,\ldots,R_r,R_{r+1}+R_{r+2}+1,\ldots,R_m)^{(i,j+1)}}\Bigg]$$

$$+\frac{(n-R_1-\cdots-R_r-r)}{(R_{r+1}+1)}\Bigg[\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)^{(i+j+2)}}$$

$$+\left\{\frac{j+2}{j+1}\right\}\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)^{(i+j+1)}}\Bigg]. \tag{2.16}$$

Proof: (2.16) can be proved by putting $s=r+1$ in (2.10).

Corollary 2.6: *For* $1\le r\le m-1,\ m\le n$ *and* $i,j\ge 0,$

$$\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j+2)}}=\frac{(j+2)}{\lambda(R_m+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}$$

$$+\left\{1-\frac{\lambda(R_m+1)}{(j+1)}\right\}\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(i,j+1)}}\Bigg]$$

$$+\frac{(n-R_1-\cdots-R_{m-1}-m+1)}{(R_m+1)}\Bigg[\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i,j+2)}}$$

$$+\left\{\frac{j+2}{j+1}\right\}\mu_{r,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(i,j+1)}}\Bigg]. \tag{2.17}$$

Proof: Corollary can be proved by putting $s=m$ in (2.10).

Theorem 2.3: *For* $1\le r<s\le m-1,\ m\le n$ *and* $i,j\ge 0,$

$$\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i+2,j)}}=\frac{(i+2)}{\lambda(R_r+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}$$

$$+\left\{1-\frac{\lambda(R_r+1)}{(i+1)}\right\}\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i+1,j)}}\Bigg]$$

$$-\frac{(n-R_1-\cdots-R_r-r)}{(R_r+1)}\Bigg[\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)^{(i+2,j)}}$$

$$+\left\{\frac{i+2}{i+1}\right\}\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)^{(i+1,j)}}\Bigg]$$

$$+\frac{(n-R_1-\cdots-R_{r-1}-r+1)}{(R_r+1)}\Bigg[\mu_{r-1,s-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,\ldots,R_m)^{(i+2,j)}}$$

$$+\left\{\frac{i+2}{i+1}\right\}\mu_{r-1,s-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,\ldots,R_m)^{(i+1,j)}}\Bigg]. \tag{2.18}$$

Proof: From (2.2), we have

$$(1+\lambda)\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i,j)}}+\lambda\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(i+1,j)}}$$

$$=(1+\lambda)E\left[X_{r:m:n}^{(R_1,\ldots,R_m)^{(i)}}X_{s:m:n}^{(R_1,\ldots,R_m)^{(j)}}\right]+\lambda E\left[X_{r:m:n}^{(R_1,\ldots,R_m)^{(i+1)}}X_{s:m:n}^{(R_1,\ldots,R_m)^{(j)}}\right]$$

$$=A(n,m-1)\iint\ldots\int_{0<x_1<\cdots<x_{r-1}<x_{r+1}<\cdots<x_m<\infty}x_s^j Z(x_{r-1},x_{r+1})f(x_1)[1-F(x_1)]^{R_1}$$

$$\times\cdots\times f(x_{r-1})[1-F(x_{r-1})]^{R_{r-1}}f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}$$

$$\times\cdots\times f(x_m)[1-F(x_m)]^{R_m}dx_1\ldots dx_{s-1}dx_{s+1}\ldots dx_m, \tag{2.19}$$

where

$$Z(x_{r-1},x_{r+1})=\int_{x_{r-1}}^{x_{r+1}}x_r^i(1+\lambda+\lambda x_r)f(x_r)[1-F(x_r)]^{R_r}dx_r. \tag{2.20}$$

On using relation (1.4) in (2.20) and integrating by parts, we get

$$Z(x_{r-1},x_{r+1})=\lambda^2\Bigg[\frac{1}{(i+1)}\Bigg(x_{r+1}^{i+1}[1-F(x_{r+1})]^{R_r+1}-x_{r-1}^{i+1}[1-F(x_{r-1})]^{R_r+1}$$

$$+(R_r+1)\int_{x_{r-1}}^{x_{r+1}}x_r^{i+1}[1-F(x_r)]^{R_r}f(x_r)dx_r\Bigg)$$

$$+\frac{1}{(i+2)}\Bigg(x_{r+1}^{i+2}[1-F(x_{r+1})]^{R_r+1}-x_{r-1}^{i+2}[1-F(x_{r-1})]^{R_r+1}$$

$$+(R_r+1)\int_{x_{r-1}}^{x_{r+1}}x_r^{i+2}[1-F(x_r)]^{R_r}f(x_r)dx_r\Bigg)\Bigg]. \tag{2.21}$$

Now by substituting the resultant expression of $Z(x_{r-1}, x_{r+1})$ from (2.21) in (2.19) and simplifying, leads to (2.18).

Corollary 2.7: *For* $2 \le r \le m-2,\ m \le n$ *and* $i, j \ge 0,$

$$\begin{aligned}
\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)(i+2,j)} &= \frac{(i+2)}{\lambda(R_r+1)}\Bigg[\left\{\frac{1+\lambda}{\lambda}\right\}\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)(i,j)} \\
&\quad + \left\{1-\frac{\lambda(R_r+1)}{(i+1)}\right\}\mu_{r,r+1:m:n}^{(R_1,\ldots,R_m)(i+1,j)}\Bigg] \\
&\quad - \frac{(n-R_1-\cdots-R_r-r)}{(R_r+1)}\Bigg[\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)(i+j+2)} \\
&\quad + \left\{\frac{i+2}{i+1}\right\}\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,\ldots,R_m)(i+j+1)}\Bigg] \\
&\quad + \frac{(n-R_1-\cdots-R_{r-1}-r+1)}{(R_r+1)}\Bigg[\mu_{r-1,r:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,\ldots,R_m)(i+2,j)} \\
&\quad + \left\{\frac{i+2}{i+1}\right\}\mu_{r-1,r:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,\ldots,R_m)(i+1,j)}\Bigg]. \qquad (2.22)
\end{aligned}$$

Proof: (2.22) can be proved by putting $s = r+1$ in (2.18).

Remark 2.1: Aggarwala and Balakrishnan (1996) have mentioned in [Remark 5, pp. 762-763] that if $R_1 = R_2 = \cdots = R_{j-1} = 0$, that is there is no censoring before the $j-th$ failure, then the first j progressively Type-II right censored order statistics are simply first j usual order statistics. Thus, for the special case $R_1 = R_2 = \cdots = R_m = 0$ so that $m = n$, in which the progressively censored order statistics becomes the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, whose single moments are denoted by $\mu_{r:n}^{(i)},\ 1 \le r \le n$ and product moments are denoted by $\mu_{r,s:n}^{(i,j)},\ 1 \le r < s \le n$, the recurrence relations established above would reduce to the following

(i) From (2.7): For $i \geq 0$,

$$\mu_{1:n}^{(i+2)} = \frac{(i+2)}{n\lambda}\left[\frac{(1+\lambda)}{\lambda}\mu_{1:n}^{(i)} + \left\{1 - \frac{n\lambda}{(i+1)}\right\}\mu_{1:n}^{(i+1)}\right].$$

(ii) From (2.14): For $2 \leq r \leq n-1$ and $i \geq 0$,

$$\mu_{r:n}^{(i+2)} = \frac{(i+2)}{\lambda(n-r+1)}\left[\frac{(1+\lambda)}{\lambda}\mu_{r:n}^{(i)} + \left\{1 - \frac{\lambda(n-r+1)}{(i+1)}\right\}\mu_{r:n}^{(i+1)}\right]$$

$$+\left[\mu_{r-1:n}^{(i+2)} + \left\{\frac{i+2}{i+1}\right\}\mu_{r-1:n}^{(i+1)}\right].$$

(iii) From (2.15): For $i \geq 0$,

$$\mu_{n:n}^{(i+2)} = \frac{(i+2)}{\lambda}\left[\frac{(1+\lambda)}{\lambda}\mu_{n:n}^{(i)} + \left\{1 - \frac{\lambda}{(i+1)}\right\}\mu_{n:n}^{(i+1)}\right] + \left[\mu_{n-1:n}^{(i+2)} + \left\{\frac{i+2}{i+1}\right\}\mu_{n-1:n}^{(i+1)}\right].$$

(iv) From (2.10): For $1 \leq r < s \leq n$ and $i, j \geq 0$,

$$\mu_{r,s:n}^{(i,j+2)} = \frac{(j+2)}{\lambda(n-s+1)}\left[\frac{(1+\lambda)}{\lambda}\mu_{r,s:n}^{(i,j)} + \left\{1 - \frac{\lambda(n-s+1)}{(j+1)}\right\}\mu_{r,s:n}^{(i,j+1)}\right]$$

$$+\left[\mu_{r,s-1:n}^{(i,j+2)} + \left\{\frac{j+2}{j+1}\right\}\mu_{r,s-1:n}^{(i,j+1)}\right].$$

(v) From (2.17): For $1 \leq r \leq n-1$ and $i, j \geq 0$,

$$\mu_{r,n:n}^{(i,j+2)} = \frac{(j+2)}{\lambda}\left[\frac{(1+\lambda)}{\lambda}\mu_{r,n:n}^{(i,j)} + \left\{1 - \frac{\lambda}{(j+1)}\right\}\mu_{r,n:n}^{(i,j+1)}\right]$$

$$+\left[\mu_{r,n-1:n}^{(i,j+2)} + \left\{\frac{j+2}{j+1}\right\}\mu_{r,n-1:n}^{(i,j+1)}\right].$$

Similarly, the remaining recurrence relations for product moments of progressively Type-II right censored order statistics can be reduced for the recurrence relation for the product moments of order statistics.

3. RECURSIVE COMPUTATIONAL ALGORITHM

Let us assume that $\mu_{1:n}$ is known. Setting $i=0$ in (2.8), we can get $\mu_{1:1:n}^{(n-1)^{(2)}}$ by the knowledge of $\mu_{1:1:n}^{(n-1)} = \mu_{1:n}$. For $i=1$, $\mu_{1:1:n}^{(n-1)^{(3)}}$ can be evaluated from $\mu_{1:1:n}^{(n-1)^{(1)}}$ and $\mu_{1:1:n}^{(n-1)^{(2)}}$. From (2.7), at $i=0$ and $m=2$, by the knowledge of $\mu_{1:2:n}^{(R_1,R_2)^{(1)}}$, one can evaluate $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$. And, for $i=1$, $\mu_{1:2:n}^{(R_1,R_2)^{(3)}}$ can be evaluated from $\mu_{1:2:n}^{(R_1,R_2)^{(1)}}$ and $\mu_{1:2:n}^{(R_1,R_2)^{(2)}}$. Similarly setting $i=0$ and $m=3$, by the knowledge $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(1)}}$, one can evaluate $\mu_{1:3:n}^{(R_1,R_2,R_3)^{(2)}}$. Therefore to evaluate the value of $\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+2)}}$, we require values of $\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(1)}}$ for all m. Now, setting $i=0$ in (2.14), $\mu_{2:3:n}^{(R_1,R_2,R_3)^{(2)}}$ can be evaluated by the knowledge of $\mu_{2:3:n}^{(R_1,R_2,R_3)^{(1)}}$. Thus for the recursive computation we have to find out $\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(1)}}$ for all r,m, by integration and then remaining by the recursive manner.

Setting $i=0$ and $m=2$ in (2.9), we can get $\mu_{1,2:2:n}^{(R_1,R_2)^{(2,j)}}$ by the knowledge of $\mu_{1,2:2:n}^{(R_1,R_2)^{(1,j)}}$. Hence to evaluate $\mu_{1,2:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}$, we need $\mu_{1,2:m:n}^{(R_1,R_2,\ldots,R_m)^{(1,j)}}$ or $\mu_{1,2:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,1)}}$. By using (2.10), one can easily evaluate $\mu_{r,r+1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}$ for all i, j by the knowledge of $\mu_{1,2:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,1)}}$.

CHAPTER IV

*MOMENT PROPERTIES OF GENERALIZED ORDER STATISTICS

1. INTRODUCTION

Recurrence relations for moments of ordered random variables such as order statistics, record values and generalized order statistics are interesting in their own rights. They usefully express the higher order moments in terms of the lower order moments and hence make the recursive computation of higher order moments easy. Furthermore, they are also useful in characterizing distributions, which is an important area, permitting the identification of population distributions from the properties of the sample [*cf.* Malik *et al.* (1988) and Al-Hussaini *et al.* (2005)].

The recurrence relations for single and product moments of specific distribution as well as for general class of distributions by order statistics and record values have received considerable interest in the past decades. There are a large number of papers available based on the moments of order statistics. One may refer to Joshi (1978, 1982), Balakrishnan and Joshi (1981a, 1981b, 1982, 1984), Khan *et al.* (1983a, 1983b), Balakrishnan and Kocherlakota (1986), Ali and Khan (1987), Khan and Khan (1987), Khan and Ali (1987), Balakrishnan *et al.* (1988), Al-Shboul and Khan (1989), Ali and Khan (1995, 1996, 1997, 1998), Kamps (1991, 1998), Balakrishnan and Sultan (1998), Ahmad (2001), Raqab (2004), Thomas and Samuel (2008) and references therein. Recurrence relations for single and product moments of some specific distributions based on record values are given by Balakrishnan *et al.* (1992), Balakrishnan *et al.* (1993), Balakrishnan and Ahsanullah (1994a, 1994b), Balakrishnan and Ahsanullah (1995), Pawlas and Szynal (1998, 1999), Raqab (2000, 2001), Minimol and Thomas (2013) among others.

Recurrence relations for single and product moments of generalized order statistics for some specific as well as for general class of distribution are

*Part of the results of this chapter are contained in Athar and Akhter (2016) investigated by several authors in literature. Cramer and Kamps (2000) derived relations for expectations of single and product moments of generalized order statistics for a general class of distributions whereas Pawlas and Szynal (2001b) obtained the recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Further, Ahmad and Fawzy (2003) derived relations for single moments of generalized order statistics for a class of doubly truncated distributions whereas Athar and Islam (2004) and Anwar *et al.* (2007) obtained the recurrence relations for single and product moments of generalized order statistics for some general class of distributions. Athar *et al.* (2007) obtained explicit expressions for ratio and inverse moments of generalized order statistics from Weibull distribution whereas Ahmad (2008) established explicit expressions and recurrence relations for single and product moments of generalized order statistics from linear exponential distribution. Mahmoud and Al-Nagar (2009) obtained the recurrence relations between the single and product moments of generalized order statistics from linear exponential distribution and characterized the linear exponential distribution based on the conditional moments of generalized order statistics. Further, Athar *et al.* (2012) obtained the moment identities of Pareto distribution based on generalized order statistics and Nayabuddin and Athar (2016) derived the recurrence relations for single and product moments of generalized order statistics from Marshall-Olkin extended Pareto distribution. For more results, one may refer to Saran and Pandey (2003), Ahmad (2007), Khan *et al.* (2007a), Athar *et al.* (2013) and references therein.

In this chapter, recurrence relations for single and product moments of generalized order statistics have been obtained when the continuous distribution function (*df*) and the probability density function (*pdf*) are functionally related as

$$f(x) = a\,x^{b}[1-F(x)]^{c}\,, \quad x \in (\alpha, \beta) \tag{1.1}$$

where a, b and c are integers.

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

Theorem 2.1: *Let* $X(r,n,m,k),\ r=1,\dots,n$ *be the* $r-th$ *gos from a continuous population with the df* $F(x)$ *and the pdf* $f(x)$ *over the support* (α,β), *if the pdf and the df are functionally related as given in* (1.1), *then for* $n\in N,\ m\in\Re$, $2\le r\le n$ *and* $p\ge b+1$,

(i) $E[X^p(r,n,m,k)]-E[X^p(r-1,n,m,k)]$

$$=\frac{p}{a\gamma_r}K^*E[X^{p-b-1}(r,n,m,k-c+1)] \tag{2.1}$$

(ii) $E[X^p(r,n,m,k)]-E[X^p(r-1,n-1,m,k)]$

$$=\frac{p}{a\gamma_1}K^*E[X^{p-b-1}(r,n,m,k-c+1)] \tag{2.2}$$

where

$$K^*=\frac{C_{r-1}}{C_{r-1}^{(n,k-c+1)}}=\prod_{i=1}^{r}\left(\frac{\gamma_i}{\gamma_i-c+1}\right)$$

and

$$C_{r-1}^{(n,k-c+1)}=\prod_{i=1}^{r}\gamma_i^{(n,k-c+1)}=\prod_{i=1}^{r}[(k-c+1)+(n-i)(m+1)].$$

Proof: In view of (1.5.6), we have

$$E[\xi\{X(r,n,m,k)\}]-E[\xi\{X(r-1,n,m,k)\}]$$

$$=\frac{C_{r-2}}{(r-1)!}\int_\alpha^\beta \xi'(x)[1-F(x)]^{\gamma_r}g_m^{r-1}(F(x))\,dx,$$

where $\xi(x)$ is a Borel measurable function of $x\in(\alpha,\beta)$.

Let $\xi(x)=x^p$, then we have

$$E[X^p(r,n,m,k)]-E[X^p(r-1,n,m,k)]$$

$$=\frac{p}{\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^{p-1}[1-F(x)]^{\gamma_r}g_m^{r-1}(F(x))\,dx.$$

Now in view of (1.1), we have

$$E[X^p(r,n,m,k)] - E[X^p(r-1,n,m,k)]$$

$$= \frac{p}{\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^{p-1}[1-F(x)]^{\gamma_r}\left\{\frac{f(x)}{a\,x^b[1-F(x)]^c}\right\}g_m^{r-1}(F(x))\,dx$$

$$= \frac{p}{a\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^{p-b-1}[1-F(x)]^{\gamma_r-c}\, f(x)g_m^{r-1}(F(x))\,dx$$

$$= \frac{p}{a\gamma_r}\frac{C_{r-1}}{C_{r-1}^{(n,k-c+1)}}\left\{\frac{C_{r-1}^{(n,k-c+1)}}{(r-1)!}\int_\alpha^\beta x^{p-b-1}[1-F(x)]^{\gamma_r^{(n,k-c+1)}-1}\, f(x)g_m^{r-1}(F(x))\,dx\right\}$$

as $\gamma_r - c = \gamma_r^{(n,k-c+1)} - 1 = [(k-c+1)+(n-r)(m+1)]-1$.

Therefore,

$$E[X^p(r,n,m,k)] - E[X^p(r-1,n,m,k)] = \frac{p}{a\,\gamma_r}K^*E[X^{p-b-1}(r,n,m,k-c+1)].$$

Hence (2.1).

Now consider the following identity (*cf*. Corollary 1.1 of Kamps, 1995, pp-116),

$$[k+(n-r-1)(m+1)]E[X^p(r,n,m,k)] + r(m+1)E[X^p(r+1,n,m,k)]$$

$$= [k+(n-1)(m+1)]E[X^p(r,n-1,m,k)], \tag{2.3}$$

on replacing r with $r-1$ in (2.3), we get

$$\gamma_r\, E[X^p(r-1,n,m,k)] + (r-1)(m+1)E[X^p(r,n,m,k)]$$

$$= \gamma_1\, E[X^p(r-1,n-1,m,k)]. \tag{2.4}$$

Now, using (2.4) in (2.1), we get

$$E[X^p(r,n,m,k)] + \frac{(r-1)(m+1)}{\gamma_r}E[X^p(r,n,m,k)] - \frac{\gamma_1}{\gamma_r}E[X^p(r-1,n-1,m,k)]$$

$$= \frac{p}{a\,\gamma_r}K^*E[X^{p-b-1}(r,n,m,k-c+1)]\ . \tag{2.5}$$

This upon rearrangement yields (2.2).

Remark 2.1: Let $m=0$ and $k=1$, then recurrence relation for single moments of order statistics is given as

(i) $$E[X_{r:n}^{p}]-E[X_{r-1:n}^{p}]=\frac{pC_{r:n}(r-1)!\,(n-c-r+1)!}{a(n-r+1)(n-c+1)!}E[X_{r:n-c+1}^{p-b-1}]. \tag{2.6}$$

(ii) $$E[X_{r:n}^{p}]-E[X_{r-1:n-1}^{p}]=\frac{pC_{r:n}(r-1)!\,(n-c-r+1)!}{a\,n(n-c+1)!}E[X_{r:n-c+1}^{p-b-1}], \tag{2.7}$$

where $C_{r:n}=\dfrac{n!}{(r-1)!(n-r)!}$

Remark 2.2: If $m=-1$, the recurrence relation for single moments of $k-th$ upper record values will be

$$E\left[X_{U^{(k)}(r)}^{p}\right]-E\left[X_{U^{(k)}(r-1)}^{p}\right]=\frac{p}{ak}\left(\frac{k}{k-c+1}\right)^{r}E\left[X_{U^{(k-c+1)}(r)}^{p-b-1}\right], \tag{2.8}$$

where $E\left[X_{U^{(k)}(r)}^{p}\right]$ is $p-th$ moment of $k-th$ upper record value.

3. EXAMPLES BASED ON THEOREM 2.1

(i) **Exponential distribution**

$$F(x)=1-e^{-\left(\frac{x-\mu}{\theta}\right)},\quad x>\mu;\quad \mu,\theta>0.$$

If we put $a=\dfrac{1}{\theta}$, $b=0$ and $c=1$ in (2.1) and (2.2), then recurrence relations for single moments of *gos* from exponential distribution is given as

$$E[X^{p}(r,n,m,k)]-E[X^{p}(r-1,n,m,k)]=\frac{p\theta}{\gamma_r}E[X^{p-1}(r,n,m,k)] \tag{3.1}$$

$$E[X^{p}(r,n,m,k)]-E[X^{p}(r-1,n-1,m,k)]=\frac{p\theta}{\gamma_1}E[X^{p-1}(r,n,m,k)] \tag{3.2}$$

Further, at $m=0$ and $k=1$ in (3.2), we get the recurrence relations for single moments of order statistics from exponential distribution as

$$E[X_{r:n}^{p}]-E[X_{r-1:n-1}^{p}]=\frac{p\theta}{n}E[X_{r:n}^{p-1}]. \tag{3.3}$$

and

$$E[X_{1:n}^{p}] = \frac{p\theta}{n} E[X_{1:n}^{p-1}]. \tag{3.4}$$

as obtained by Khan *et al.* (1983a) and Balakrishnan and Sultan (1998) for non-truncated case.

At $m = -1$ in (3.1), we obtain the recurrence relation for single moments of $k-th$ upper record values

$$E\left[X_{U^{(k)}(r)}^{p}\right] - E\left[X_{U^{(k)}(r-1)}^{p}\right] = \frac{p\theta}{k} E\left[X_{U^{(k)}(r)}^{p-1}\right]. \tag{3.5}$$

(ii) **Rayleigh distribution**

$$F(x) = 1 - e^{-\theta x^2}, \quad 0 < x < \infty; \quad \theta > 0.$$

If we put $a = 2\theta$, $b = 1$ and $c = 1$ in (2.1) and (2.2), then recurrence relations for single moments of *gos* from Rayleigh distribution is given as

$$E[X^{p}(r,n,m,k)] - E[X^{p}(r-1,n,m,k)] = \frac{p}{2\theta\gamma_r} E[X^{p-2}(r,n,m,k)] \tag{3.6}$$

$$E[X^{p}(r,n,m,k)] - E[X^{p}(r-1,n-1,m,k)] = \frac{p}{2\theta\gamma_1} E[X^{p-2}(r,n,m,k)] \tag{3.7}$$

Further, at $m = 0$ and $k = 1$ in (3.6) and (3.7), we get the recurrence relations for single moments of order statistics from Rayleigh distribution, as

$$E[X_{r:n}^{p}] - E[X_{r-1:n}^{p}] = \frac{p}{2\theta(n-r+1)} E[X_{r:n}^{p-2}]. \tag{3.8}$$

$$E[X_{r:n}^{p}] - E[X_{r-1:n-1}^{p}] = \frac{p}{2n\theta} E[X_{r:n}^{p-2}]. \tag{3.9}$$

At $m = -1$ in (3.6), we obtain the recurrence relation for single moments of $k-th$ upper record values from the Rayleigh distribution, as

$$E\left[X_{U^{(k)}(r)}^{p}\right] - E\left[X_{U^{(k)}(r-1)}^{p}\right] = \frac{p}{2\theta k} E\left[X_{U^{(k)}(r)}^{p-2}\right]. \tag{3.10}$$

(iii) **Weibull distribution**

$$F(x) = 1 - e^{-\theta x^{\nu}}, \quad 0 < x < \infty; \quad \theta, \nu > 0.$$

If we put $a = \nu\theta$, $b = \nu - 1$ and $c = 1$ in (2.1) and (2.2), then recurrence relations for single moments of *gos* from the Weibull distribution is given as

$$E[X^{p}(r,n,m,k)] - E[X^{p}(r-1,n,m,k)] = \frac{p}{\nu\theta\gamma_r} E[X^{p-\nu}(r,n,m,k)] \qquad (3.11)$$

$$E[X^{p}(r,n,m,k)] - E[X^{p}(r-1,n-1,m,k)] = \frac{p}{\nu\theta\gamma_1} E[X^{p-\nu}(r,n,m,k)]. \qquad (3.12)$$

The relation in (3.11) was given by Ahmad and Fawzy (2003).

Further, at $m = 0$ and $k = 1$ in (3.12), we get the recurrence relation for single moments of order statistics from the Weibull distribution

$$E[X_{r:n}^{p}] - E[X_{r-1:n-1}^{p}] = \frac{p}{n\nu\theta} E[X_{r:n}^{p-\nu}] \qquad (3.13)$$

as obtained by Khan *et al.* (1983a) for non-truncated case.

At $m = -1$ in (3.11), we obtain the recurrence relation for single moments of $k-th$ upper record values from the Weibull distribution, as

$$E\left[X_{U^{(k)}(r)}^{p}\right] - E\left[X_{U^{(k)}(r-1)}^{p}\right] = \frac{p}{\nu\theta k} E\left[X_{U^{(k)}(r)}^{p-\nu}\right]. \qquad (3.14)$$

Further, at $k = 1$ in (3.14), we obtain the recurrence relation for single moments of upper record values as obtained by Ahmad and Fawzy (2003).

(iv) **Pareto distribution**

$$F(x) = 1 - \left(\frac{\theta}{x}\right)^{\nu}, \quad \theta < x < \infty; \quad \theta, \nu > 0.$$

If we put $a = \nu$, $b = -1$ and $c = 1$ in (2.1) and (2.2), then recurrence relations for single moments of *gos* from the Pareto distribution is given as

$$E[X^p(r,n,m,k)] = \left(\frac{\nu\gamma_r}{\nu\gamma_r - p}\right) E[X^p(r-1,n,m,k)], \quad \nu\gamma_r \neq p \tag{3.15}$$

$$E[X^p(r,n,m,k)] = \left(\frac{\nu\gamma_1}{\nu\gamma_1 - p}\right) E[X^p(r-1,n-1,m,k)], \quad \nu\gamma_1 \neq p \tag{3.16}$$

The relation in (3.15) was obtained by Athar *et al.* (2012), whereas the relation in (3.16) by Ahmad and Fawzy (2003).

Further, at $m = 0$ and $k = 1$ in (3.16), we get the recurrence relation for single moments of order statistics from the Pareto distribution

$$E[X_{r:n}^p] = \left(\frac{n\nu}{n\nu - p}\right) E[X_{r-1:n-1}^p], \; n\nu \neq p \tag{3.17}$$

as obtained by Malik (1966), Khan *et al.* (1983a) and Balakrishnan and Sultan (1998) for non-truncated case.

At $m = -1$ in (3.15), we obtain the recurrence relation for single moments of $k-th$ upper record values from the Pareto distribution

$$E\left[X_{U^{(k)}(r)}^p\right] = \left(\frac{\nu k}{\nu k - p}\right) E\left[X_{U^{(k)}(r-1)}^p\right], \; \nu k \neq p \tag{3.18}$$

as obtained by Pawlas and Szynal (1999).

At $k = 1$ in (3.18), we can get the recurrence relation for single moments of upper record values as obtained by Ahmad and Fawzy (2003).

4. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

Theorem 4.1: *Let* $X(r,n,m,k)$ *and* $X(s,n,m,k)$, $1 \le r < s \le n-1$ *be the* $r-th$ *and* $s-th$ *gos from a continuous population with the df* $F(x)$ *and the pdf* $f(x)$ *over the support* (α, β), *if the pdf and the df are functionally related as given in* (1.1), *then for* $n \in N$, $m \in \Re$, $1 \le r < s \le n-1$, $n \ge 2$ *and* $p = 1,2,\ldots$; $q \ge b+1$,

$$E[X^{p}(r,n,m,k)X^{q}(s,n,m,k)]-E[X^{p}(r,n,m,k)X^{q}(s-1,n,m,k)]$$

$$=\frac{q}{a\gamma_s}K^{**}E[X^{p}(r,n,m,k)X^{q-b-1}(s,n,m,k-c+1)] \tag{4.1}$$

where

$$K^{**}=\frac{C_{s-1}}{C_{s-1}^{(n,k-c+1)}}=\prod_{i=1}^{s}\left(\frac{\gamma_i}{\gamma_i-c+1}\right)$$

and

$$C_{s-1}^{(n,k-c+1)}=\prod_{i=1}^{s}\gamma_i^{(n,k-c+1)}=\prod_{i=1}^{s}[(k-c+1)+(n-i)(m+1)].$$

Proof: In view of (1.5.9), we have

$$E[X^{p}(r,n,m,k)X^{q}(s,n,m,k)]-E[X^{p}(r,n,m,k)X^{q}(s-1,n,m,k)]$$

$$=\frac{q}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{x}^{\beta}x^{p}y^{q-1}\,[1-F(x)]^{m}f(x)g_m^{r-1}(F(x))$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s}\,dydx. \tag{4.2}$$

Now in view of (1.1), we get

$$=\frac{q}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{x}^{\beta}x^{p}y^{q-1}\,[1-F(x)]^{m}f(x)g_m^{r-1}(F(x))$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s}\left\{\frac{f(y)}{a\,y^{b}[1-F(y)]^{c}}\right\}dydx.$$

Hence (4.1) can be established after noting that $\gamma_s-c=\gamma_s^{(n,k-c+1)}-1$ and $C_{s-1}=\gamma_s C_{s-2}$.

Remark 4.1: Let $m=0$ and $k=1$, the recurrence relation for product moments of order statistics is

$$E[X_{r:n}^{p}X_{s:n}^{q}]-E[X_{r:n}^{p}X_{s-1:n}^{q}]$$

$$=\frac{qC_{s:n}(s-1)!(n-c-s+1)!}{a(n-s+1)(n-c+1)!}E[X_{r:n}^{p}X_{s:n-c+1}^{q-b-1}]. \tag{4.3}$$

Remark 4.2: Let $m=-1$, the recurrence relation for product moments of $k-th$ upper record values will be

$$E\left[X^{p}_{U^{(k)}(r)}\,X^{q}_{U^{(k)}(s)}\right]-E\left[X^{p}_{U^{(k)}(r)}\,X^{q}_{U^{(k)}(s-1)}\right]$$

$$=\frac{q}{a\,k}\left(\frac{k}{k-c+1}\right)^{s}E\left[X^{p}_{U^{(k)}(r)}\,X^{q-b-1}_{U^{(k-c+1)}(s)}\right]. \tag{4.4}$$

5. EXAMPLES BASED ON THEOREM 4.1

(i) **Exponential distribution**

$$F(x)=1-e^{-\left(\frac{x-\mu}{\theta}\right)},\quad x>\mu;\quad \mu,\theta>0.$$

If we put $a=\frac{1}{\theta}$, $b=0$ and $c=1$ in (4.1), then recurrence relation for product moments of *gos* from exponential distribution is given as

$$E[X^{p}(r,n,m,k)X^{q}(s,n,m,k)]-E[X^{p}(r,n,m,k)X^{q}(s-1,n,m,k)]$$

$$=\frac{q\theta}{\gamma_s}E[X^{p}(r,n,m,k)X^{q-1}(s,n,m,k)]\,. \tag{5.1}$$

Further, at $m=0$ and $k=1$ in (5.1), we get the recurrence relation for product moments of order statistics from exponential distribution

$$E[X^{p}_{r:n}X^{q}_{s:n}]-E[X^{p}_{r:n}X^{q}_{s-1:n-1}]=\frac{q\theta}{(n-s+1)}E[X^{p}_{r:n}X^{q-1}_{s:n}]. \tag{5.2}$$

as obtained by Khan *et al.* (1983b) and Balakrishnan and Sultan (1998) for non-truncated case.

At $m=-1$ in (5.1), we obtain the recurrence relation for product moments of $k-th$ upper record values from exponential distribution as

$$E\left[X^{p}_{U^{(k)}(r)}X^{q}_{U^{(k)}(s)}\right]-E\left[X^{p}_{U^{(k)}(r)}X^{q}_{U^{(k)}(s-1)}\right]=\frac{q\theta}{k}E\left[X^{p}_{U^{(k)}(r)}X^{q-1}_{U^{(k)}(s)}\right] \tag{5.3}$$

(ii) **Rayleigh distribution**

$$F(x) = 1 - e^{-\theta x^2}, \quad 0 < x < \infty; \quad \theta > 0.$$

If we put $a = 2\theta$, $b = 1$ and $c = 1$ in (4.1), then recurrence relation for product moments of *gos* from the Rayleigh distribution is given as

$$E[X^p(r,n,m,k)X^q(s,n,m,k)] - E[X^p(r,n,m,k)X^q(s-1,n,m,k)]$$

$$= \frac{q}{2\theta\gamma_s} E[X^p(r,n,m,k)X^{q-2}(s,n,m,k)]. \tag{5.4}$$

Further, at $m = 0$ and $k = 1$ in (5.4), we get the recurrence relation for product moments of order statistics from the Rayleigh distribution, as

$$E[X_{r:n}^p X_{s:n}^q] - E[X_{r:n}^p X_{s-1:n}^q] = \frac{q}{2\theta(n-s+1)} E[X_{r:n}^p X_{s:n}^{q-2}]. \tag{5.5}$$

At $m = -1$ in (5.4), we obtain the recurrence relation for product moments of $k-th$ upper record values from the Rayleigh distribution as

$$E\left[X_{U^{(k)}(r)}^p X_{U^{(k)}(s)}^q\right] - E\left[X_{U^{(k)}(r)}^p X_{U^{(k)}(s-1)}^q\right]$$

$$= \frac{p}{2\theta k} E\left[X_{U^{(k)}(r)}^p X_{U^{(k)}(s)}^{q-2}\right]. \tag{5.6}$$

(iii) **Weibull distribution**

$$F(x) = 1 - e^{-\theta x^\nu}, \quad 0 < x < \infty; \quad \theta, \nu > 0.$$

If we put $a = \nu\theta$, $b = \nu - 1$ and $c = 1$ in (4.1), then recurrence relation for product moments of *gos* from the Weibull distribution is given as

$$E[X^p(r,n,m,k)X^q(s,n,m,k)] - E[X^p(r,n,m,k)X^q(s-1,n,m,k)]$$

$$= \frac{q}{\nu\theta\gamma_s} E[X^p(r,n,m,k)X^{q-\nu}(s,n,m,k)] \tag{5.7}$$

Further, at $m=0$ and $k=1$ in (5.7), we get the recurrence relation for product moments of order statistics from the Weibull distribution

$$E[X_{r:n}^{p}X_{s:n}^{q}]-E[X_{r:n}^{p}X_{s-1:n}^{q}]=\frac{q}{\nu\theta(n-s+1)}E[X_{r:n}^{p}X_{s:n}^{q-\nu}] \tag{5.8}$$

as obtained by Khan *et al.* (1983b) for non-truncated case.

At $m=-1$ in (5.7), we obtain the recurrence relation for product moments of $k-th$ upper record values from the Weibull distribution as

$$E\left[X_{U^{(k)}(r)}^{p}X_{U^{(k)}(s)}^{q}\right]-E\left[X_{U^{(k)}(r)}^{p}X_{U^{(k)}(s-1)}^{q}\right]$$

$$=\frac{q}{\nu\theta k}E\left[X_{U^{(k)}(r)}^{p}X_{U^{(k)}(s)}^{q-\nu}\right]. \tag{5.9}$$

(iv) **Pareto distribution**

$$F(x)=1-\left(\frac{\theta}{x}\right)^{\nu},\quad \theta<x<\infty;\quad \theta,\nu>0.$$

If we put $a=\nu$, $b=-1$ and $c=1$ in (4.1), then recurrence relation for product moments of *gos* from the Pareto distribution is given as

$$E[X^{p}(r,n,m,k)X^{q}(s,n,m,k)]$$

$$=\left(\frac{\nu\gamma_s}{\nu\gamma_s-q}\right)E[X^{p}(r,n,m,k)X^{q}(s-1,n,m,k)],\quad \nu\gamma_s\neq q \tag{5.10}$$

The relation in (5.10) was given by Athar *et al.* (2012).

Further, at $m=0$ and $k=1$ in (5.10), we get the recurrence relation for product moments of order statistics from the Pareto distribution

$$E[X_{r:n}^{p}X_{s:n}^{q}]=\left(\frac{\nu(n-s+1)}{\nu(n-s+1)-q}\right)E[X_{r:n}^{p}X_{s-1:n}^{q}],\quad \nu(n-s+1)\neq q \tag{5.11}$$

as obtained by Khan *et al*. (1983b) and Balakrishnan and Sultan (1998) for non-truncated case.

At $m=-1$ in (5.10), we obtain the recurrence relation for product moments of $k-th$ upper record values from the Pareto distribution

$$E\left[X^{p}_{U^{(k)}(r)}X^{q}_{U^{(k)}(s)}\right]=\left(\frac{\nu k}{\nu k-q}\right)E\left[X^{p}_{U^{(k)}(r)}X^{q}_{U^{(k)}(s-1)}\right],\ \ \nu k\neq q. \tag{5.12}$$

*CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS BASED ON ORDER STATISTICS

1. INTRODUCTION

Characterization of distributions through the properties of conditional expectations of order statistics have been studied by several authors. Various approaches are available in literature. For detailed survey one may refer to Khan and Ali (1987), Nagaraja (1988b), Khan and Abu-Salih (1989), Balasubramanian and Beg (1992), Franco and Ruiz (1995, 1997), Balasubramanian and Dey (1997), López-Blázquez and Moreno-Rebello (1997), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998), Khan and Abouammoh (2000), Athar *et al.* (2003), Khan and Athar (2004) and references therein.

In this chapter, an attempt is made to characterize a general form of distribution $F(x)=ah(x)+b,\ x\in(\alpha,\beta)$ through conditional expectation of $p-th$ power of difference of functions of two order statistics, conditioned on a pair of non-adjacent order statistics.

2. CHARACTERIZATION THEOREM

Theorem 2.1: *Let* $X_{r:n},\ r=1,2,\ldots,n$ *be the* $r-th$ *order statistic from a continuous population with the df* $F(x)$ *and the pdf* $f(x)$ *over the support* (α,β). *Then for* $1\le l<i<s\le n,\ l=r,r+1$

$$E[(h(X_{i:n})-h(X_{l:n}))^p \mid X_{l:n}=x, X_{s:n}=y]=g_{l,i,s,p}(x,y)$$

$$=\frac{\Gamma(p+i-l)\Gamma(s-l)}{\Gamma(p+s-l)\Gamma(i-l)}\{h(y)-h(x)\}^p \qquad (2.1)$$

if and only if

$$F(x)=ah(x)+b \qquad (2.2)$$

---

*Part of the results of this chapter are contained in Athar and Akhter (2015b) *where* a, b *are so chosen that* $F(x)$ *is a df and* $h(x)$ *is a monotonic, continuous and differentiable function of* x *and* p *is a positive integer.*

Proof: To prove the necessary part, we have

$$E[(h(X_{i:n})-h(X_{r:n}))^p \mid X_{r:n}=x, X_{s:n}=y]=g_{r,i,s,p}(x,y)$$

$$=\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^p\left[\frac{F(t)-F(x)}{F(y)-F(x)}\right]^{i-r-1}$$

$$\times\left[1-\frac{F(t)-F(x)}{F(y)-F(x)}\right]^{s-i-1}\frac{f(t)}{F(y)-F(x)}dt,$$

in view of (1.2.14).

Let

$$\frac{F(t)-F(x)}{F(y)-F(x)}=z,$$

which implies that

$$(h(t)-h(x))^p=z^p(h(y)-h(x))^p.$$

Then, we have

$$E[(h(X_{i:n})-h(X_{r:n}))^p \mid X_{r:n}=x, X_{s:n}=y]=g_{r,i,s,p}(x,y)$$

$$=\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_0^1 (h(y)-h(x))^p z^{p+i-r-1}(1-z)^{s-i-1}dz,$$

and hence the necessary part.

To prove the sufficiency part, consider

$$E[(h(X_{i:n})-h(X_{r:n}))^p \mid X_{r:n}=x, X_{s:n}=y]=g_{r,i,s,p}(x,y)$$

or,

$$\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^p[F(t)-F(x)]^{i-r-1}[F(y)-F(t)]^{s-i-1}f(t)dt$$

$$=g_{r,i,s,p}(x,y)[F(y)-F(x)]^{s-r-1}. \tag{2.3}$$

Differentiating (2.3) *w.r.t.* x, we get

$$-ph'(x)\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^{p-1}[F(t)-F(x)]^{i-r-1}$$

$$\times[F(y)-F(t)]^{s-i-1}f(t)dt$$

$$-\frac{(i-r-1)\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^{p}[F(t)-F(x)]^{i-r-2}$$

$$\times[F(y)-F(t)]^{s-i-1}f(t)f(x)dt$$

$$=\frac{\partial}{\partial x}\left\{g_{r,i,s,p}(x,y)\right\}[F(y)-F(x)]^{s-r-1}$$

$$-(s-r-1)\,g_{r,i,s,p}(x,y)[F(y)-F(x)]^{s-r-2}f(x) \qquad (2.4)$$

or,

$$-ph'(x)\,g_{r,i,s,p-1}(x,y)-(s-r-1)\,g_{r+1,i,s,p}(x,y)\frac{f(x)}{[F(y)-F(x)]}$$

$$=\frac{\partial}{\partial x}g_{r,i,s,p}(x,y)-(s-r-1)g_{r,i,s,p}(x,y)\frac{f(x)}{[F(y)-F(x)]}. \qquad (2.5)$$

After, rearranging the terms of (2.5), we get

$$\frac{f(x)}{[F(y)-F(x)]}=\frac{1}{(s-r-1)}\frac{ph'(x)g_{r,i,s,p-1}(x,y)+\frac{\partial}{\partial x}g_{r,i,s,p}(x,y)}{g_{r,i,s,p}(x,y)-g_{r+1,i,s,p}(x,y)}. \qquad (2.6)$$

Now consider,

$$ph'(x)g_{r,i,s,p-1}(x,y)+\frac{\partial}{\partial x}g_{r,i,s,p}(x,y)$$

$$=ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r)}{\Gamma(p+s-r-1)\,\Gamma(i-r)}$$

$$-ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)}$$

$$= ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(s-r)}{\Gamma(i-r)}\left[\frac{\Gamma(p+i-r-1)}{\Gamma(p+s-r-1)}-\frac{\Gamma(p+i-r)}{\Gamma(p+s-r)}\right]$$

$$= ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r)}{\Gamma(p+s-r-1)\,\Gamma(i-r)}\left[1-\frac{p+i-r-1}{p+s-r-1}\right]$$

$$= p(s-i)h'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)} \tag{2.7}$$

and

$$g_{r,i,s,p}(x,y)-g_{r+1,i,s,p}(x,y)$$

$$=\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)}$$

$$-\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r-1)}{\Gamma(p+s-r-1)\,\Gamma(i-r-1)}$$

$$= p(s-i)\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r-1)}{\Gamma(p+s-r)\,\Gamma(i-r)}. \tag{2.8}$$

Therefore in view of (2.6), we have

$$\frac{f(x)}{[F(y)-F(x)]}=\frac{h'(x)}{[h(y)-h(x)]}.$$

Hence the theorem.

Remark 2.1: At $p=1$, (2.1) reduces to

$$h(\hat{X}_{i:n})=E[h(X_{i:n})\,|\,X_{r:n}=x, X_{s:n}=y]$$

$$=\frac{(s-i)h(x)+(i-r)h(y)}{(s-r)},\ i=r+1,r+2,\ldots,s-1 \tag{2.9}$$

or equivalently

$$E[h(X_{r+i:n})\,|\,X_{r:n}=x, X_{s:n}=y]$$

$$=\frac{(m-i+1)h(x)+ih(y)}{(m+1)},\ \ m=s-r-1,\ \ 1\le r<s\le n,\ \ i=1,2,\ldots,m. \tag{2.10}$$

as obtained by Khan and Athar (2004).

Further, at $s = n+1$, $X_{n+1:n} = y = \beta$ and $m = n - r$, then we have

$$E[h(X_{i:n}) \mid X_{r:n} = x] = \frac{(n-i+1)h(x) + (i-r)h(\beta)}{(n-r+1)}, \; i = r+1,\, r+2, \ldots, n \quad (2.11)$$

or,

$$E[h(X_{r+i:n}) \mid X_{r:n} = x] = \frac{(n-r-i+1)h(x) + ih(\beta)}{(n-r+1)},$$

$$1 \le r < s \le n, \; i = 1,2,\ldots,n-r. \quad (2.12)$$

and at $r = 0$, $X_{0:n} = x = \alpha$ and $m = s-1$, we get

$$E[h(X_{i:n}) \mid X_{s:n} = y] = \frac{(s-i)h(\alpha) + ih(y)}{s},$$

as given by Khan and Abu-Salih (1989), Franco and Ruiz (1997), Khan and Abouammoh (2000) and Khan and Athar (2004).

Remark 2.2: At $p = 1$, $i = r+1$ and $s = r+2$ in Theorem 2.1, we have

$$E[h(X_{r+1:n}) \mid X_{r:n} = x, X_{r+2:n} = y] = E[h(X) \mid x \le X \le y]$$

$$= \frac{h(x) + h(y)}{2} \quad (2.13)$$

which may also be expressed as,

$$\frac{1}{(s-r-1)} \sum_{i=r+1}^{s-1} E[h(X_{i:n}) \mid X_{r:n} = x, X_{s:n} = y] = \frac{h(x)+h(y)}{2} \quad (2.14)$$

as obtained by Balasubramanian and Beg (1992).

3. EXAMPLES

(i) **Power function distribution**

$$F(x) = \theta^{-\nu} x^{\nu}, \; 0 < x \le \theta; \; \theta, \nu > 0.$$

Then $F(x)$ is given by (2.2) with $a = \theta^{-\nu}$, $b = 0$ and $h(x) = x^{\nu}$.

(ii) **Pareto distribution**

$$F(x) = 1 - \theta^{\nu} x^{-\nu}, \ \ \theta \le x < \infty; \ \theta, \nu > 0.$$

Then $F(x)$ is given by (2.2) with $a = -\theta^{\nu}$, $b = 1$ and $h(x) = x^{-\nu}$.

(iii) **Exponential distribution**

$$F(x) = 1 - e^{-\theta x}, \ \ 0 < x < \infty; \theta > 0.$$

Then $F(x)$ is given by (2.2) with $a = -1$, $b = 1$ and $h(x) = e^{-\theta x}$.

(iv) **Rayleigh distribution**

$$F(x) = 1 - e^{-\theta x^2}, \ \ 0 < x < \infty; \theta > 0.$$

Then $F(x)$ is given by (2.2) with $a = -1$, $b = 1$ and $h(x) = e^{-\theta x^2}$.

(v) **Weibull distribution**

$$F(x) = 1 - e^{-\theta x^{\nu}}, \ \ 0 < x < \infty; \theta, \nu > 0.$$

Then $F(x)$ is given by (2.2) with $a = -1$, $b = 1$ and $h(x) = e^{-\theta x^{\nu}}$.

(vi) **Inverse Weibull distribution**

$$F(x) = e^{-\theta x^{-\nu}}, \ \ 0 < x < \infty; \theta, \nu > 0.$$

Then $F(x)$ is given by (2.2) with $a = 1$, $b = 0$ and $h(x) = e^{-\theta x^{-\nu}}$.

(vii) **Extreme value distribution**

$$F(x) = 1 - e^{-e^{x}}, \ \ -\infty < x < \infty.$$

Then $F(x)$ is given by (2.2) with $a = -1$, $b = 1$ and $h(x) = e^{-e^{x}}$.

(viii) **Cauchy distribution**

$$F(x) = \frac{1}{\pi}\tan^{-1} x + \frac{1}{2}, \ -\infty < x < \infty.$$

Then $F(x)$ is given by (2.2) with $a = \frac{1}{\pi}$, $b = \frac{1}{2}$ and $h(x) = \tan^{-1} x$.

(ix) **Gumbel distribution**

$$F(x) = e^{-e^{-x}}, \ -\infty < x < \infty.$$

Then $F(x)$ is given by (2.2) with $a = 1, b = 0$ and $h(x) = e^{-e^{-x}}$.

(x) **Burr type XII distribution**

$$F(x) = 1 - (1 + \theta x^{\gamma})^{-\nu}, \ 0 < x < \infty; \theta, \gamma, \nu > 0.$$

Then $F(x)$ is given by (2.2) with $a = -1, b = 1$ and $h(x) = (1 + \theta x^{\gamma})^{-\nu}$.

Similarly, characterization results for some other distributions may also be obtained with proper choice of a, b and $h(x)$. One may refer to Khan and Athar (2004).

4. NUMERICAL ILLUSTRATIONS

Let us consider here the example of Mann and Fertig (1973), who gave the failure times of airplane components for a life test in which 13 components were placed on test, with the test terminating after the $10-th$ failure. Failure times in hours are recorded as below

0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00.

Now, if we calculate the expected values of the randomly selected failure times of airplane components $\hat{X}_{2:10}, \hat{X}_{4:10}, \hat{X}_{6:10}$ and $\hat{X}_{8:10}$ by using (2.9), we get

					Distributions				
					Power ($\nu=1$)	**Pareto** ($\nu=1$)	**Exponential** ($\theta=1$)	**Rayleigh** ($\theta=1$)	**Weibull** ($\nu=0.5,\theta=0.5$)
n	r	i	s	$X_{i:n}$	$\hat{X}_{i:n}$	$\hat{X}_{i:n}$	$\hat{X}_{i:n}$	$\hat{X}_{i:n}$	$\hat{X}_{i:n}$
10	1	2	3	0.50	0.55	0.35	0.50	0.59	0.48
	1	2	4	0.50	0.48	0.30	0.42	0.53	0.40
	1	2	6	0.50	0.44	0.26	0.36	0.48	0.35
	1	2	8	0.50	0.44	0.25	0.34	0.44	0.33
	1	2	10	0.50	0.53	0.25	0.33	0.41	0.33
	1	4	5	1.00	1.05	0.58	0.91	0.99	0.91
	2	4	6	1.00	0.92	0.73	0.83	0.86	0.84
	3	4	5	1.00	1.10	1.06	1.07	1.07	1.08
	3	4	9	1.00	1.15	0.99	1.02	0.98	1.06
	3	4	10	1.00	1.18	0.98	1.01	0.96	1.07
	1	6	7	1.33	1.32	0.77	1.16	1.20	1.19
	2	6	8	1.33	1.34	0.96	1.15	1.11	1.21
	3	6	8	1.33	1.41	1.26	1.31	1.25	1.35
	5	6	7	1.33	1.43	1.42	1.42	1.42	1.43
	5	6	10	1.33	1.66	1.49	1.49	1.40	1.57
	1	8	9	1.76	2.22	1.09	1.76	1.45	1.97
	2	8	9	1.76	2.21	1.59	1.85	1.48	2.05
	6	8	9	1.76	2.11	1.93	1.95	1.69	2.04
	7	8	9	1.76	2.02	1.90	1.91	1.74	1.97
	7	8	10	1.76	2.03	1.83	1.84	1.67	1.94
$\boldsymbol{MSE}=\frac{1}{20}\sum_{i=1}^{20}(\hat{X}_i - X_i)^2$					0.0444	0.0759	0.0146	0.0160	0.0262

The Mean Square Error (MSE) of the expected values of the failure times of airplane components using exponential distribution is minimum. Thus, we can conclude that the exponential distribution gives the best fit.

It may be noted that only those parameter(s) value of the distributions are arbitrarily set which are involved in $h(x)$ while characterizing the distribution.

CHAPTER VI

CHARACTERIZATION OF CONTINUOUS PROBABILITY DISTRIBUTIONS THROUGH RECORD VALUES

1. INTRODUCTION

Characterization of distributions through conditional expectation of record values have been considered among others by Nagaraja (1977, 1988b), Franco and Ruiz (1996, 1997), López-Blázquez and Moreno-Rebollo (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000), Wu and Lee (2001), Athar *et al.* (2003), Gupta and Ahsanullah (2004), Wu (2004), Khan *et al.* (2010b), Khan and Athar (2010) and Noor and Athar (2014). In these papers the authors have assumed the linearity of regression while characterizing the distribution functions.

Bairamov *et al.* (2005) characterized the exponential distribution and its monotone transforms through conditional expectation conditioned on a pair of adjacent record values. Further, Yanev *et al.* (2008) extended the results of Bairamov *et al.* (2005) and considered the case for non-adjacent covariates.

In this chapter, a family of continuous distributions of the form $F(x) = 1 - e^{-ah(x)},\ x \in (\alpha, \beta)$ have been characterized through conditional expectation of $p-th$ power of difference of functions of two upper record values, conditioned on a pair of non-adjacent records.

2. CHARACTERIZATION THEOREM

Theorem 2.1: *Let* $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ *be the record values from a continuous population with the df* $F(x)$ *and the pdf* $f(x)$ *over the support* (α, β). *Then for* $1 \le l < i < s \le n,\ l = r, r+1$

$$E\left[\left(h\left(X_{U(i)}\right) - h\left(X_{U(l)}\right)\right)^p \mid X_{U(l)} = x, X_{U(s)} = y\right] = \xi_{l,i,s,p}(x, y)$$

$$= \frac{\Gamma(p+i-l)\Gamma(s-l)}{\Gamma(p+s-l)\Gamma(i-l)}\{h(y) - h(x)\}^p \qquad (2.1)$$

if and only if

$$\overline{F}(x) = e^{-ah(x)}, \ a \neq 0 \tag{2.2}$$

where $h(x)$ *is a continuous, differentiable and non-decreasing function of* x *and* p *is a positive integer and* $\overline{F}(x) = 1 - F(x)$.

Proof: First we shall prove the necessary part.

In view of (1.4.6), we have

$$E\left[\left(h(X_{U(i)}) - h(X_{U(r)})\right)^p \mid X_{U(r)} = x, X_{U(s)} = y\right] = \xi_{r,i,s,p}(x,y)$$

$$= \frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)} \int_x^y (h(t)-h(x))^p \left[\frac{-\ln \overline{F}(t) + \ln \overline{F}(x)}{-\ln \overline{F}(y) + \ln \overline{F}(x)}\right]^{i-r-1}$$

$$\times \left[1 - \frac{-\ln \overline{F}(t) + \ln \overline{F}(x)}{-\ln \overline{F}(y) + \ln \overline{F}(x)}\right]^{s-i-1} \frac{1}{-\ln \overline{F}(y) + \ln \overline{F}(x)} \frac{f(t)}{\overline{F}(t)} dt \ .$$

If we let

$$\left[\frac{-\ln \overline{F}(t) + \ln \overline{F}(x)}{-\ln \overline{F}(y) + \ln \overline{F}(x)}\right] = z,$$

then

$$(h(t) - h(x))^p = z^p (h(y) - h(x))^p.$$

Thus, we have

$$E\left[\left(h(X_{U(i)}) - h(X_{U(r)})\right)^p \mid X_{U(r)} = x, X_{U(s)} = y\right]$$

$$= \frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)} \int_0^1 (h(y)-h(x))^p z^{p+i-r-1} (1-z)^{s-i-1} dz,$$

and hence the necessary part.

To prove the sufficiency part, we have

$$E\left[\left(h(X_{U(i)}) - h(X_{U(r)})\right)^p \mid X_{U(r)} = x, X_{U(s)} = y\right] = \xi_{r,i,s,p}(x,y)$$

or,

$$\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^p[-\ln\overline{F}(t)+\ln\overline{F}(x)]^{i-r-1}$$
$$\times[-\ln\overline{F}(y)+\ln\overline{F}(t)]^{s-i-1}\frac{f(t)}{\overline{F}(t)}dt$$

$$=\xi_{r,i,s,p}(x,y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}. \tag{2.3}$$

Differentiating (2.3) *w.r.t.* x, we get

$$-ph'(x)\frac{\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\int_x^y (h(t)-h(x))^{p-1}[-\ln\overline{F}(t)+\ln\overline{F}(x)]^{i-r-1}$$
$$\times[-\ln\overline{F}(y)+\ln\overline{F}(t)]^{s-i-1}\frac{f(t)}{\overline{F}(t)}dt$$

$$-\frac{(i-r-1)\Gamma(s-r)}{\Gamma(i-r)\Gamma(s-i)}\frac{f(x)}{\overline{F}(x)}\int_x^y (h(t)-h(x))^p[-\ln\overline{F}(t)+\ln\overline{F}(x)]^{i-r-2}$$
$$\times[-\ln\overline{F}(y)+\ln\overline{F}(t)]^{s-i-1}\frac{f(t)}{\overline{F}(t)}dt$$

$$=\frac{\partial}{\partial x}\{\xi_{r,i,s,p}(x,y)\}[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}$$
$$-(s-r-1)\,\xi_{r,i,s,p}(x,y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-2}\frac{f(x)}{\overline{F}(x)} \tag{2.4}$$

or,

$$-ph'(x)\,\xi_{r,i,s,p-1}(x,y)-(s-r-1)\,\xi_{r+1,i,s,p}(x,y)\frac{1}{[-\ln\overline{F}(y)+\ln\overline{F}(x)]}\frac{f(x)}{\overline{F}(x)}$$

$$=\frac{\partial}{\partial x}\xi_{r,i,s,p}(x,y)-(s-r-1)\xi_{r,i,s,p}(x,y)\frac{1}{[-\ln\overline{F}(y)+\ln\overline{F}(x)]}\frac{f(x)}{\overline{F}(x)}. \tag{2.5}$$

Rearranging the terms of (2.5), we get

$$\frac{1}{[-\ln\overline{F}(y)+\ln\overline{F}(x)]}\frac{f(x)}{\overline{F}(x)}$$

$$=\frac{1}{(s-r-1)}\frac{ph'(x)\xi_{r,i,s,p-1}(x,y)+\dfrac{\partial}{\partial x}\xi_{r,i,s,p}(x,y)}{\xi_{r,i,s,p}(x,y)-\xi_{r+1,i,s,p}(x,y)}. \tag{2.6}$$

Consider,

$$ph'(x)\xi_{r,i,s,p-1}(x,y)+\frac{\partial}{\partial x}\xi_{r,i,s,p}(x,y)$$

$$= ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r)}{\Gamma(p+s-r-1)\,\Gamma(i-r)}$$

$$- ph'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)},$$

$$= p(s-i)h'(x)\{h(y)-h(x)\}^{p-1}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)} \tag{2.7}$$

and

$$\xi_{r,i,s,p}(x,y)-\xi_{r+1,i,s,p}(x,y)$$

$$=\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r)\,\Gamma(s-r)}{\Gamma(p+s-r)\,\Gamma(i-r)}$$

$$-\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r-1)}{\Gamma(p+s-r-1)\,\Gamma(i-r-1)}$$

$$= p(s-i)\{h(y)-h(x)\}^{p}\frac{\Gamma(p+i-r-1)\,\Gamma(s-r-1)}{\Gamma(p+s-r)\,\Gamma(i-r)}. \tag{2.8}$$

Therefore in view of (2.6), we have

$$\frac{1}{[-\ln\bar{F}(y)+\ln\bar{F}(x)]}\frac{f(x)}{\bar{F}(x)}=\frac{h'(x)}{[h(y)-h(x)]}$$

implying that

$$F(x)=1-e^{-ah(x)},\ \ a\neq 0.$$

Hence the theorem.

Corollary 2.1: *Under the conditions as stated in Theorem 2.1,*

$$h(\hat{X}_{U(i)}) = E[h(X_{U(i)}) \mid X_{U(r)} = x, X_{U(s)} = y]$$

$$= \frac{(s-i)h(x) + (i-r)h(y)}{(s-r)} \tag{2.9}$$

and consequently

$$E[h(X_{U(r+1)}) \mid X_{U(r)} = x, X_{U(r+2)} = y] = \frac{h(x)+h(y)}{2} \tag{2.10}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \;\; a \neq 0,$$

where $h(x)$ *is a monotonic and differentiable function of* x.

Proof: Expression (2.9) can be obtained at $p=1$ from Theorem 2.1 as established by Khan and Khan (2009).

3. EXAMPLES

Several well known distributions can be characterized using Theorem 2.1 with proper choice of a and $h(x)$ as given in Table 3.1 or one may refer to Noor and Athar (2014).

Table 3.1: Examples based on the $F(x) = 1 - e^{-ah(x)}, \; a \neq 0.$

Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{a}$	$\boldsymbol{h(x)}$
Exponential	$1-e^{-\theta x}, \; 0<x<\infty; \;\; \theta>0$	θ	x
Rayleigh	$1-e^{-\theta x^2}, 0<x<\infty; \;\; \theta>0$	θ	x^2
Weibull	$1-e^{-\theta x^\nu}, \; 0<x<\infty; \;\; \theta,\nu>0$	θ	x^ν
Pareto	$1-\left(\frac{x}{\theta}\right)^{-\nu}, \; \theta \le x<\infty; \;\; \theta,\nu>0$	ν	$\log\left(\frac{x}{\theta}\right)$
Lomax	$1-(1+x)^{-\nu}, 0<x<\infty; \;\; \nu>0$	ν	$\log(1+x)$
Beta of the I kind	$1-(1-x)^{\theta}, \; 0<x<1; \;\; \theta>0$	$-\theta$	$\log(1-x)$

Beta of the II kind	$1-(1+x)^{-1}$, $0<x<\infty$	1	$\log(1+x)$
Extreme value I	$1-\exp(-e^{x})$, $-\infty<x<\infty$	1	e^{x}
Log logistic	$1-(1+\theta x^{\nu})^{-1}$, $0<x<\infty;\ \theta,\nu>0$	1	$\log(1+\theta x^{\nu})$
Burr type XII	$1-(1+\theta x^{\nu})^{-\gamma}$, $0<x<\infty;\ \nu,\theta,\gamma>0$	γ	$\log(1+\theta x^{\nu})$

4. NUMERICAL ILLUSTRATIONS

Consider the survival times in (days) of a group of lung cancer patients [Soliman and Al-Aboud (2008)]

6.96, 9.30, 6.96, 7.24, 9.30, 4.90, 8.42, 6.05, 10.18, 6.82, 8.58, 7.77, 11.94, 11.25, 12.94, 12.94.

If only the upper record values have been observed, then these are

6.96, 9.30, 10.18, 11.94, 12.94.

Now, if we calculate the expected values of randomly selected upper record values at $p=1$, using Corollary 2.1, then we get

				Distributions			
				Exponential	Rayleigh	Weibull ($\nu=0.5$)	Pareto ($\theta=1$)
r	i	s	$X_{U(i)}$	$\hat{X}_{U(i)}$	$\hat{X}_{U(i)}$	$\hat{X}_{U(i)}$	$\hat{X}_{U(i)}$
1	2	3	9.30	8.57	8.72	8.49	8.41
		4	9.30	8.62	8.93	8.47	8.33
		5	9.30	8.44	8.44	8.28	8.13
	3	4	10.18	10.28	10.54	10.13	9.97
		5	10.18	9.95	10.39	9.72	9.49
	4	5	11.94	11.45	11.73	11.27	11.08
2	3	4	10.18	10.62	10.70	10.58	10.54
		5	10.18	10.51	10.65	10.44	10.38
	4	5	11.94	11.73	11.85	11.66	11.59
3	4	5	11.94	11.56	11.64	11.52	11.48
$\boldsymbol{MSE}=\frac{1}{10}\sum_{i=1}^{10}(\hat{X}_{U(i)}-X_{U(i)})^{2}$				0.2529	0.2020	0.3531	0.4865

The Mean Square Error (MSE) of the expected values of upper records, using Rayleigh distribution is minimum. Thus, we can conclude that the Rayleigh distribution gives the best fit.

It may be noted that only those parameter(s) value of the distributions are arbitrarily set which are involved in $h(x)$ while characterizing the distribution.

CHAPTER VII

*CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH GENERALIZED ORDER STATISTICS AND DUAL GENERALIZED ORDER STATISTICS

1. INTRODUCTION

The problem of characterization of distributions has attracted the interest of many researchers. The conditional expectation of order statistics are extensively used in characterizing probability distributions. Khan and Abu-Salih (1989) have characterized some general form of distributions through conditional expectation of function of order statistics, conditioned on adjacent order statistic. Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) when the conditioning may not be adjacent. Khan *et al.* (2006) established characterizing relationships for the distributions through generalized order statistics whereas Khan *et al.* (2007b) established characterizing results for order statistics when conditioning is on a pair of order statistics. Further, Samuel (2008) characterized the distributions considered by Khan and Abu-Salih (1989) for the generalized order statistics. Khan *et al.* (2010a) characterized several distributions through conditional expectation of function of dual generalized order statistics.

For more detailed survey on characterization one may refer to Franco and Ruiz (1995, 1997), López-Blázquez and Moreno-Rebollo (1997), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998, 2000), Keseling (1999), Wu and Ouyang (1996), Khan and Alzaid (2004), Khan and Athar (2004), Athar *et al.* (2003) and references therein.

In this chapter, results of Samuel (2008) have been extended and several continuous distributions are characterized when conditioning is not on adjacent values. Characterization theorems based on generalized order statistics and dual generalized order statistics are presented. Further, some deductions for order statistics and record values are also discussed and several examples are listed.

---

*Part of the results of this chapter are contained in Noor *et al.* (2015)

2. CHARACTERIZATION THEOREMS

Theorem 2.1: *Let X be a random variable with absolutely continuous df $F(x)$ and pdf $f(x)$ over the support (α,β) and $h(x)$ be a monotonic, continuous and differentiable function of x, then for $1 \le l < s \le n$, $l = r, r+1$*

$$E[h\{X(s,n,m,k)\} \mid X(l,n,m,k) = x] = g_{s|l}(x) = h(x) + \frac{1}{a}\sum_{j=l}^{s-1}\frac{1}{\gamma_{j+1}},$$

$$a \ne 0 \text{ and } \gamma_{j+1} \ne 0 \quad (2.1)$$

if and only if

$$\overline{F}(x) = e^{-ah(x)}, \; a \ne 0, \quad (2.2)$$

where $X(r,n,m,k)$, $r = 1,2,\ldots,n$ is the $r-th$ gos and $\overline{F}(x) = 1 - F(x)$.

Proof: To prove the necessary part, for $s \ge r+1$,

$$E[h\{X(s,n,m,k)\} \mid X(r,n,m,k) = x]$$

$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^\beta h(y)\left[1 - \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_s - 1}\frac{f(y)}{\overline{F}(x)}dy. \quad (2.3)$$

Set

$$u = \frac{\overline{F}(y)}{\overline{F}(x)} = \frac{e^{-ah(y)}}{e^{-ah(x)}},$$

which implies

$$h(y) = h(x) - \frac{1}{a}\ln u.$$

Then the *RHS* of (2.3) reduces to

$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^1\left(h(x) - \frac{1}{a}\ln u\right)\left(1-u^{m+1}\right)^{s-r-1}u^{\gamma_s - 1}du.$$

Let $u^{m+1} = t,$ then we get

$$E[h\{X(s,n,m,k)\} \mid X(r,n,m,k) = x]$$

$$= h(x) - \frac{\prod_{j=r+1}^{s} \gamma_j}{(s-r-1)!\,(m+1)^{s-r+1}} \frac{1}{a} \int_0^1 \ln t \; t^{\frac{\gamma_s}{m+1} - 1} (1-t)^{s-r-1} dt$$

$$= h(x) - \frac{\prod_{j=r+1}^{s} \gamma_j}{(s-r-1)!(m+1)^{s-r+1}} \frac{1}{a} B\left(\frac{\gamma_s}{m+1}, s-r\right) \left[\psi\left(\frac{\gamma_s}{m+1}\right) - \psi\left(\frac{\gamma_r}{m+1}\right)\right],$$

where

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x),$$

$$\psi(x-n) - \psi(x) = -\sum_{k=1}^{n} \left(\frac{1}{x-k}\right) \tag{2.4}$$

[*cf*. Gradshteyn and Ryzhik, 2007, pp-540, 905]

and $B(a,b)$ is the complete beta function.

Therefore,

$$E\left[h\{X(s,n,m,k)\} \mid X(r,n,m,k) = x\right] = h(x) + \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{\gamma_{j+1}}.$$

To prove the sufficiency part, let

$$E\left[h\{X(s,n,m,k)\} \mid X(r,n,m,k) = x\right] = g_{s|r}(x).$$

Therefore,

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\beta h(y) \left[\left(\overline{F}(x)\right)^{m+1} - \left(\overline{F}(y)\right)^{m+1}\right]^{s-r-1} \left[\overline{F}(y)\right]^{\gamma_s - 1} f(y) dy$$

$$= g_{s|r}(x) \left[\overline{F}(x)\right]^{\gamma_{r+1}}.$$

Differentiating both the sides with respect to x and adjusting the terms, we get

$$\frac{f(x)}{\overline{F}(x)} = -\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} \qquad \text{[Khan } et\ al., \text{ 2006]}$$

$$= -\frac{1}{\gamma_{r+1}} \frac{h'(x)}{\left[h(x) + \frac{1}{a}\sum_{j=r+1}^{s-1} \frac{1}{\gamma_{j+1}} - h(x) - \frac{1}{a}\sum_{j=r}^{s-1} \frac{1}{\gamma_{j+1}} \right]}$$

$$= a\,h'(x)\,.$$

implying that

$$F(x) = 1 - e^{-ah(x)}.$$

Remark 2.1: At $s = r+1$ and $l = r$ in (2.2), we have

$$E\big[h\{X(r+1,n,m,k)\} \mid X(r,n,m,k) = x\big] = h(x) + \frac{1}{a}\frac{1}{\gamma_{r+1}}$$

as obtained by Samuel (2008).

Remark 2.2: For order statistics (at $m = 0$, $k = 1$), characterization result is given by Khan and Abouammoh (2000) and Wu and Ouyang (1996).

$$E[h(X_{s:n}) \mid X_{r:n} = x] = h(x) + \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{n-j}$$

Further, at $s = r+1$, we get

$$E[h(X_{r+1:n}) \mid X_{r:n} = x] = h(x) + \frac{1}{a(n-r)}$$

as obtained by Khan and Abu-Salih (1989).

Remark 2.3: Several well known distributions may be characterized using Theorem 2.1 with proper choice of a and $h(x)$ as given in Table 6.3.1.

Theorem 2.2: *Let* X *be a random variable with absolutely continuous df* $F(x)$ *and pdf* $f(x)$ *over the support* (α,β) *and* $h(x)$ *be a monotonic, continuous and differentiable function of* x, *then for* $1 \le r < l \le n$, $l = s-1$, s

$$E\left[h\{X(r,n,m,k)\} \mid X(l,n,m,k) = y\right] = g_{r|l}(y) = h(y) + \frac{1}{a}\sum_{j=r}^{l-1}\frac{1}{j}$$

$$a \ne 0 \text{ and } j \ne 0 \qquad (2.5)$$

if and only if

$$1-[\overline{F}(x)]^{m+1} = e^{-ah(x)}, \; m \ne -1 \qquad (2.6)$$

and

$$\overline{F}(x) = \exp\left[-\exp\left(\frac{h(q)-h(x)}{\delta}\right)\right], \; m=-1, \; \delta = \frac{1}{a} \ne 0, \; x<q, \qquad (2.7)$$

where $-\log \overline{F}(q) = 1$.

***Proof*:** To prove (2.6) implies (2.5), let

$$E\left[h\{X(r,n,m,k)\} \mid X(s,n,m,k) = y\right] = g_{r|s}(y),$$

or

$$g_{r|s}(y) = \frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\int_{\alpha}^{y} h(x)\left[\frac{1-(\overline{F}(x))^{m+1}}{1-(\overline{F}(y))^{m+1}}\right]^{r-1}$$

$$\times\left[1-\frac{1-(\overline{F}(x))^{m+1}}{1-(\overline{F}(y))^{m+1}}\right]^{s-r-1}\frac{(\overline{F}(x))^{m}}{1-(\overline{F}(y))^{m+1}}\, f(x)dx.$$

Let

$$u = \frac{1-(\overline{F}(x))^{m+1}}{1-(\overline{F}(y))^{m+1}} = \frac{e^{-ah(x)}}{e^{-ah(y)}}, \text{ then}$$

$$g_{r|s}(y) = \frac{(s-1)!}{(r-1)!(s-r-1)!}\int_{0}^{1}\left(h(y)-\frac{1}{a}\ln u\right)u^{r-1}(1-u)^{s-r-1}du$$

$$= h(y) - \frac{1}{a}\frac{(s-1)!}{(r-1)!(s-r-1)!}\int_0^1 \ln u\, u^{r-1}(1-u)^{s-r-1}du$$

$$= h(y) - \frac{1}{a}\frac{(s-1)!}{(r-1)!(s-r-1)!}B(r, s-r)[\psi(r) - \psi(s)].$$

Now in view of (2.4), we have

$$g_{r|s}(y) = h(y) + \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{j}.$$

Now to prove (2.5) implies (2.6), we have

$$E\big[h\{X(r,n,m,k)\} \mid X(s,n,m,k) = y\big] = g_{r|s}(y),$$

or

$$\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\int_\alpha^y h(x)\big[\overline{F}(x)\big]^m \big[1-(\overline{F}(x))^{m+1}\big]^{r-1}$$

$$\times\big[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}\big]^{s-r-1} f(x)dx$$

$$= g_{r|s}(y)\big[1-(\overline{F}(y))^{m+1}\big]^{s-1}.$$

Differentiating both the sides with respect to y and rearranging the terms, we get

$$\frac{(m+1)f(y)[\overline{F}(y)]^m}{1-[\overline{F}(y)]^{m+1}} = \frac{1}{(s-1)}\frac{g'_{r|s}(y)}{[g_{r|s-1}(y) - g_{r|s}(y)]} \qquad \text{[Khan } et\ al.\text{, 2006]}$$

$$= \frac{h'(y)}{(s-1)\left[h(y) + \frac{1}{a}\sum_{j=r}^{s-2}\frac{1}{j} - h(y) - \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{j}\right]}.$$

That is,

$$\frac{(m+1)f(y)[\overline{F}(y)]^m}{1-[\overline{F}(y)]^{m+1}} = -ah'(y) \tag{2.8}$$

implying that

$$1-[\overline{F}(x)]^{m+1} = e^{-ah(x)}, \quad m \neq -1$$

and hence the (2.6).

Now to prove (2.7), we note that $-\log \overline{F}(x)$, $\alpha < x < \beta$ is a non-decreasing function in $(0,\infty)$, therefore there exists a q such that $-\log \overline{F}(q) = 1$.

Taking the limit as $m \to -1$ in the *LHS* of (2.8), we get.

$$\frac{f(y)}{\overline{F}(y)} \frac{1}{[-\log \overline{F}(y)]} = -ah'(y)$$

or

$$\log\left[-\log \overline{F}(x)\right] = -\int_x^q a\, h'(y)dy, \;\; q \in (\alpha, \beta),$$

which gives,

$$\overline{F}(x) = \exp\left[-\exp\left(\frac{h(q)-h(x)}{\delta}\right)\right], \;\; m = -1, \;\; \delta = \frac{1}{a} \neq 0.$$

Remark 2.4: For order statistics (at $m = 0, k = 1$), characterization results based on linear regression were obtained by Khan and Abu-Salih (1989), Khan and Abouammoh (2000) and for record values (at $m = -1, k = 1$) by Franco and Ruiz (1997).

Remark 2.5: Table III of Franco and Ruiz (1997) may be used to obtain more characterizing results based on Theorem 2.2.

Theorem 2.3: *Let X be a random variable with absolutely continuous df $F(x)$ and pdf $f(x)$ over the support (α, β) and $h(x)$ be a monotonic, continuous and differentiable function of x, then for $1 \le l < s \le n$, $l = r$, $r+1$*

$$E\left[h\{X_d(s,n,m,k)\} \mid X_d(l,n,m,k) = x\right] = g_{s|l}(x) = h(x) + \frac{1}{a}\sum_{j=l}^{s-1}\frac{1}{\gamma_{j+1}},$$

$$a \neq 0 \text{ and } \gamma_{j+1} \neq 0 \qquad (2.9)$$

if and only if

$$F(x) = e^{-ah(x)}, \quad a \neq 0, \qquad (2.10)$$

where $X_d(r,n,m,k)$, $r = 1, 2, \ldots, n$ is the $r-th$ dgos.

Proof: Necessary part can be proved on the lines of Theorem 2.1.

To prove the sufficiency part, let

$$g_{s|r}(x) = E\left[h\{X_d(s,n,m,k)\} \mid X_d(r,n,m,k) = x\right]$$

Therefore, we have

$$g_{s|r+1}(x) - g_{s|r}(x) = -\frac{1}{a}\frac{1}{\gamma_{r+1}}.$$

Now, in view of Khan *et al.* (2010a), we get

$$\frac{f(x)}{\overline{F}(x)} = -ah'(x),$$

implying that

$$F(x) = e^{-ah(x)}.$$

Hence the theorem.

Remark 2.6: Several well known distributions may be characterized using Theorem 2.3 with proper choice of a and $h(x)$ as given in Table 2.1.

Table 2.1: Examples based on the $F(x) = e^{-ah(x)}$, $a \neq 0$.

Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{a}$	$\boldsymbol{h(x)}$
Inverse Weibull	$e^{-\theta x^{-\nu}}$, $0 \le x < \infty$; $\nu, \theta > 0$	θ	$x^{-\nu}$
Burr Type II	$\left(1+e^{-x}\right)^{-\nu}$, $-\infty < x < \infty$; $\nu > 0$	ν	$\log\left(1+e^{-x}\right)$
Burr Type III	$\left(1+x^{-c}\right)^{-\nu}$, $0 \le x < \infty$; $c, \nu > 0$	ν	$\log\left(1+x^{-c}\right)$
Burr Type IV	$\left(1+\left(\frac{c-x}{x}\right)^{1/c}\right)^{-\nu}$, $0 \le x \le c$; $\nu > 0$	ν	$\log\left(1+\left(\frac{c-x}{x}\right)^{1/c}\right)$
Burr Type V	$\left(1+ce^{-\tan x}\right)^{-\nu}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$; $\nu > 0$	ν	$\log\left(1+ce^{-\tan x}\right)$
Burr Type VI	$\left(1+ce^{-\nu \sinh x}\right)^{-\nu}$, $-\infty < x < \infty$; $\nu > 0$	ν	$\log\left(1+ce^{-\nu \sinh x}\right)$
Burr Type VII	$2^{-\nu}(1+\tanh x)^{\nu}$, $-\infty < x < \infty$; $\nu > 0$	$-\nu$	$\log\left(\frac{1+\tanh x}{2}\right)$

Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^x\right)^\nu$, $-\infty < x < \infty;\ \nu > 0$	$-\nu$	$\log\left(\frac{2}{\pi}\tan^{-1}e^x\right)$
Burr Type X	$\left(1-e^{-x^2}\right)^\nu$, $0 \le x < \infty;\ \nu > 0$	$-\nu$	$\log\left(1-e^{-x^2}\right)$
Burr Type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^\nu$, $0 \le x \le 1;\ \nu > 0$	$-\nu$	$\log\left(x-\frac{1}{2\pi}\sin 2\pi x\right)$
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$, $-\infty < x < \infty$	-1	$\log\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x\right)$

Theorem 2.4: *Under the conditions as stated in Theorem 2.2 and for* $1 \le r < l \le n,\ l = s-1, s$

$$E[h\{X_d(r,n,m,k)\} \mid X_d(l,n,m,k) = y] = g_{r|l}(y) = h(y) + \frac{1}{a}\sum_{j=r}^{l-1}\frac{1}{j},$$

$$a \ne 0 \text{ and } j \ne 0 \quad (2.11)$$

if and only if

$$1-[F(x)]^{m+1} = e^{-ah(x)},\ m \ne -1 \quad (2.12)$$

and

$$F(x) = \exp\left[-\exp\left(\frac{h(x)-h(p)}{\delta}\right)\right],\ m=-1,\ \delta = \frac{1}{a} \ne 0,\ y > p, \quad (2.13)$$

where $-\log F(p) = 1$.

***Proof*:** Theorem can be established on the lines of Theorem 2.2.

REFERENCES

Aggarwala, R. and Balakrishnan, N. (1996): Recurrence relations for single and product moments of progressive Type-II right censored order statistics from exponential and truncated exponential distributions. *Ann. Inst. Statist. Math.*, **48**(4), 757-771.

Ahmad, A.A. (2001): Moments of order statistics from doubly truncated continuous distributions and characterizations, *Statistics*, **35**, 479-494.

Ahmad, A.A. (2007): Recurrence relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution. *J. Egypt. Math. Soc.*, **15**(1), 117-128.

Ahmad, A.A. (2008): Single and product moments of generalized order statistics from linear exponential distribution. *Comm. Statist. Theory Methods*, **37**(8), 1162-1172.

Ahmad, A.A. and Fawzy, M.A. (2003): Recurrence relations for single moments of generalized order statistics from doubly truncated distribution. *J. Statist. Plann. Inference*, **117**, 241-249.

Ahsanullah, M. (1995): *Record Statistics.* Nova Science Publishers, New York.

Ahsanullah, M. (2004): A characterization of the uniform distribution by dual generalized order statistics. *Comm. Statist. Theory Methods*, **33**, 2921-2928.

Ahsanullah, M. and Wesolowski, J. (1998): Linearity of best predictors for non-adjacent record values, *Sankhyā, Ser. B*, **60**, 221-227.

Al-Hussaini, E.K., Ahmad, A.A. and Al-Kashif, M.A. (2005): Recurrence relations for moment and conditional moment generating function of generalized order statistics. *Metrika*, **61**, 199-220.

Ali Mousa, M.A.M. and Jaheen, Z.F. (2002): Bayesian prediction for progressively censored data from the Burr model. *Statist. Papers*, **43**, 587-593.

Ali, M.A. and Khan, A.H. (1995): Ratio and inverse moments of order statistics from Burr distribution. *J. Indian Soc. Probab. Statist.*, **2**, 97-102.

Ali, M.A. and Khan, A.H. (1996): Ratio and inverse moments of order statistics from Weibull and exponential distribution. *J. Appl. Statist. Sci.*, **4**, 1-8.

Ali, M.A. and Khan, A.H. (1997): Recurrence relations for expectations of a function of single order statistic from a general class of distributions. *J. Indian Statist. Assoc*, **35**, 1-9.

Ali, M.A. and Khan, A.H. (1998): Recurrence relations for expected values of certain functions of two order statistics. *Metron*, **LVI**, 107-119.

Ali, M.M. and Khan, A.H. (1987): On order statistics from the log-logistic distribution. *J. Statist. Plann. Inference*, **17**, 103-108.

Al-Shboul, Q.M. and Khan, A.H. (1989): Moments of order statistics from doubly truncated log-logistic distribution. *Pakistan J. Statist. Ser. B*, **3**, 299-307.

Anwar, Z., Athar, H. and Khan R.U. (2007): Expectation identities based on recurrence relations of functions of generalized order statistics. *J. Statist. Res.*, **41**(2), 93-102.

Arnold, B.C. and Balakrishnan, N. (1989): *Relations, Bounds and Approximations for order statistics.* Lecture Notes in Statistics, **53**, Springer-Verlag, New York.

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992): *A First Course in Order Statistics.* John Wiley & Sons, New York.

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998): *Records.* John Wiley & Sons, New York.

Athar, H. and Akhter, Z. (2015a): Recurrence relations between moments of progressive Type-II right censored order statistics from doubly truncated Weibull distribution. *Statist. Papers*, DOI: 10.1007/s00362-015-0658-2.

Athar, H. and Akhter, Z. (2015b): Some characterization of continuous distributions based on order statistics. *Int. J. Comp. Theo. Stat.*, **2**(1), 31-36.

Athar, H. and Akhter, Z. (2016): Some results on relations between moments of generalized order statistics. *J. Adv. Res. Appl. Math.*, In press.

Athar, H. and Islam, H.M. (2004): Recurrence relations for single and product moments of generalized order statistics from a general class of distribution. *Metron*, **LXII**(3), 327-337.

Athar, H., Akhter, Z. and Saran, J. (2014): Moments of progressive Type-II right censored order statistics from Lindley distribution. *Statistics Research Letters*, **3**(1), 1-6.

Athar, H., Anwar, Z. and Khan, R.U. (2008): On recurrence relations for the expectations of function of lower generalized order statistics. *Pakistan. J. Statist.*, **24**(2), 111-122.

Athar, H., Islam, H.M. and Yaqub, M. (2007): On ratio and inverse moments of generalized order statistics from Weibull distribution. *J. Appl. Statist. Sci.*, **16**(1), 37-46.

Athar, H., Khwaja, S.K. and Nayabuddin (2012): Expectation identities of Pareto distribution based on generalized order statistics and its characterization. *American Journal of Applied Mathematics and Mathematical Sciences*, **1**(1), 23-29.

Athar, H., Nayabuddin and Khwaja, S.K. (2013): Recurrence relations for single and product moments of generalized order statistics from doubly truncated Makeham distribution and its characterization. *J. Statist. Res.*, **47**(1), 63-71.

Athar, H., Yaqub, M. and Islam, H.M. (2003): On characterization of distributions through linear regression of record values and order statistics. *Aligarh J. Statist.*, **23**, 97-105.

Bairamov, I. (2007): Preface. *Comm. Statist. Theory Methods*, **36**, 1271-1272.

Bairamov, I., Ahsanullah, M. and Pakes, A.G. (2005): A characterization of continuous distributions via regression on pairs of record values. *Aust. N. Z. J. Stat.*, **47**, 543-547.

Balakrishnan, N. (2007): Progressive censoring methodology: an appraisal (with discussion). *Test*, **16**, 211-296.

Balakrishnan, N. and Aggarwala, R. (2000): *Progressive Censoring: Theory, Methods and Applications,* Birkhauser, Boston.

Balakrishnan, N. and Ahsanullah, M. (1994a): Recurrence relations for single and product moments of record values from generalized Pareto distribution. *Comm. Statist. Theory Methods*, **23**(10), 2841-2852.

Balakrishnan, N. and Ahsanullah, M. (1994b): Relations for single and product moments of record values from Lomax distribution. *Sankhyā* , *Ser. B*, **56**, 140-146.

Balakrishnan, N. and Ahsanullah, M. (1995): Relations for single and product moments of record values from exponential distribution. *J. Appl. Statist. Sci.*, **2**(1), 73-88.

Balakrishnan, N. and Cohen, A.C. (1991): *Order statistics and Inference: Estimation Methods*. Academic Press, San Diego.

Balakrishnan, N. and Cramer, E. (2014): *The Art of Progressive Censoring: Applications to Reliability and Quality*, Birkhauser, Boston.

Balakrishnan, N. and Joshi, P.C. (1981a): A note on order statistics from Weibull distribution. *Scand. Actur. J.*, 121-122.

Balakrishnan, N. and Joshi, P.C. (1981b): Moments of order statistics from doubly truncated Power function distribution. *Aligarh J. Statist.*, **1,** 98-105.

Balakrishnan, N. and Joshi, P.C. (1982): Moments of order statistics from doubly truncated Pareto distribution. *J. Indian Statist. Assoc.*, **20,** 109-117.

Balakrishnan, N. and Joshi, P.C. (1984): Product moments of order statistics from the doubly truncated exponential distribution. *Naval Res. Logist. Quart.*, **31**, 27-31.

Balakrishnan, N. and Kocherlakota, S. (1986): On the moments of order statistics from the doubly truncated logistic distribution. *J. Statist. Plann. Inference*, **13**, 117-129.

Balakrishnan, N. and Saleh, H.M. (2011): Relations for moments of progressively Type-II censored order statistics from half-logistic distribution with applications to inference. *Comput. Statist. Data Anal.*, **55**, 2775-2792.

Balakrishnan, N. and Saleh, H.M. (2012): Relations for moments of progressively Type-II censored order statistics from log-logistic distribution with applications to inference. *Comm. Statist. Theory Methods*, **41**, 880-906.

Balakrishnan, N. and Saleh, H.M. (2013): Recurrence relations for single and product moments of progressively Type-II censored order statistics from a generalized half-logistic distribution with applications to inference. *J. Stat. Comput. Simul.*, **83**(9), 1704-1721.

Balakrishnan, N. and Sandhu, R.A. (1995): A simple simulation algorithm for generating progressive Type-II censored samples. *Amer. Statist.*, **49**, 229-230.

Balakrishnan, N., Ahsanullah, M. and Chan, P.S. (1992): Relations for single and product moments of record values from Gumbel distribution. *Statist. Probab. Lett.*, **15**, 223-227.

Balakrishnan, N., Chan, P.S. and Ahsanullah, M. (1993): Recurrence relations for moments of record values from generalized extreme value distribution. *Comm. Statist. Theory Methods*, **22**(5), 1471-1482.

Balakrishnan, N. and Sultan, K.S. (1998): Recurrence relations and identities for moments of order statistics, In: N. Balakrishnan and C.R. Rao (Eds) Handbook of statistics, vol. 16. Order statistics: Theory and Methods, North-Holland, Amsterdam, 149-228.

Balakrishnan, N., Al-Hussaini, E.K. and Saleh, H.M. (2011): Recurrence relations for moments of progressively censored order statistics from logistic distribution with applications to inference. *J. Statist. Plann. Inference*, **141**, 17-30.

Balakrishnan, N., Cramer, E., Kamps, U. and Schenk, N. (2001): Progressive Type-II censored order statistics from exponential distributions. *Statistics*, **35**, 537-556.

Balakrishnan, N., Malik, H.J. and Ahmed, S.E. (1988): Recurrence relations and identities for moments of order statistics, II: Specific continuous distributions. *Comm. Statist. Theory Methods*, **17**(8), 2657-2694.

Balasubramanian K. and Beg, M.I. (1992): Distributions determined by conditioning on a pair of order statistic. *Metrika*, **39**, 107-112.

Balasubramanian, K. and Dey, A. (1997): Distribution characterized through conditional expectations. *Metrika*, **45**, 189-196.

Barakat, H.M. (2007): Limit theory of generalized order statistics. *J. Statist. Plann. Inference*, **137**, 1-11.

Barakat, H.M. and El-Adll, M.E. (2009): Asymptotic theory of extreme dual generalized order statistics. *Statist. Probab. Lett.*, **79**, 1252-1259.

Barakat, H.M. and El-Adll, M.E. (2012): On the limit distribution of lower extreme generalized order statistics. *Proc. Indian Acad. Sci. Math.*, **122**, 297-311.

Barakat, H.M., El-Adll, M.E. and Aly, A.E. (2011): Exact prediction intervals for future exponential lifetime based on random generalized order statistics. *Comput. Math. Appl.*, **61**, 1366-1378.

Burkschat, M., Cramer, E. and Kamps, U. (2003): Dual generalized order statistics. *Metron*, **LXI**(1), 13-26.

Chandler, K.N. (1952): The distribution and frequency of record values. *J. Roy. Statist. Soc. Ser. B*, **14**, 220-228.

Cohen, A.C. (1963): Progressively censored samples in life testing. *Technometrics*, **5**, 327-339.

Cohen, A.C. (1966): Life testing and early failure. *Technometrics*, **8**, 539-549.

Cohen, A.C. (1975): Multi-censored sampling in the three parameter Weibull distribution. *Technometrics*, **17**, 347-351.

Cohen, A.C. (1976): Progressively censored sampling in the three parameter log-normal distribution. *Technometric*, **18**, 99-103.

Cohen, A.C. (1991): *Truncated and Censored Samples: Theory and Applications*. Marcel Dekker, New York.

Cohen, A.C. and Whitten, B.J. (1982): Modified maximum likelihood and modified moment estimators for the three-parameter Weibull distribution. *Comm. Statist. Theory Methods*, **11**, 2631-2656.

Cohen, A.C. and Whitten, B.J. (1988): *Parameter estimation in reliability and life span models*. Marcel Dekker, New York

Cramer, E. and Kamps, U. (2000): Relations for expectations of functions of generalized order statistics. *J. Statist. Plann. Inference*, **89**, 79-89.

Cramer, E. and Kamps, U. (2003): Marginal distributions of sequential and generalized order statistics. *Metrika*, **58**, 293-310.

David, H.A. and Nagaraja, H.N. (2003): *Order Statistics*. John Wiley & Sons, New York.

Dembińska, A. and Wesolowski, J. (1998): Linearity of regression for non-adjacent order statistics. *Metrika*, **48**, 215-222.

Dembińska, A. and Wesolowski, J. (2000): Linearity of regression for non adjacent record values. *J. Statist. Plann. Inference*, **90**, 195-205.

Dziubdziela, W. and Kopociński, B. (1976): Limiting properties of the $k-th$ record values. *Appl. Math.*, **15**, 187-190.

Franco, M. and Ruiz, J.M. (1995): On characterization of continuous distributions with adjacent order statistics. *Statistics*, **26**, 375-385.

Franco, M. and Ruiz, J.M. (1996): On characterization of continuous distributions by conditional expectation of record values. *Sankhy$\bar{a}$, Ser. A*, **58**, 135-141.

Franco, M. and Ruiz, J.M. (1997): On characterizations of distributions by expected values of order statistics and record values with gap. *Metrika*, **45**, 107-119.

Galambos, J. (1987): *The Asymptotic Theory of Extreme Order Statistics.* Second edition, Kreiger, Malabar, Florida.

Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008): Lindley distribution and its application. *Math. Comput. Simulation*, **78**, 493-506.

Glick, N. (1978): Breaking records and breaking boards. *Amer. Math. Monthly*, **85**, 2-26.

Gradshteyn, I.S. and Ryzhik, I.M. (2007): *Tables of Integrals, Series and Products*. Edited by Jeffrey, A. and Zwillinger, D. 7th Ed., Academic Press, New York.

Greenberg, B.G. and Sarhan, A.E. (1958): Applications of order statistics to health data. *Am. J. Public Health.* **48**(10), 1388-1394.

Gupta, R.C. and Ahsanullah, M. (2004): Some characterization results based on the conditional expectation of a function of non-adjacent order statistics (record value). *Ann. Inst. Statist. Math.*, **56**(4), 721-732.

Hartley, H.O. (1942): The range in random samples. *Biometrika*, **32**, 334-348.

Herd G.R. (1960): Estimation of reliability from incomplete data. *Proceedings Sixth National Symposium on Reliability and Quality Control*, 202-217.

Herd, G.R. (1956): *Estimation of the parameters of a population from a multi-censored sample*. Ph.D. Thesis, Iowa State College, Ames, Iowa.

Herd, G.R. (1957): Estimation of reliability functions. *Proceedings Third National Symposium on Reliability and Quality Control*, 113-122.

Hussain, E. (2006): *The non-linear functions of order statistics and their properties in selected probability models*. Ph.D. Thesis. Department of Statistics, University of Karachi, Pakistan.

Johnson, N.L., Kotz, S. and Balakrishnan N. (1994): *Continuous Univariate Distributions*, Vol. I, 2nd edn. John Wiley & Sons, New York.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995): *Continuous Univariate Distributions*, Vol II, 2nd edn. John Wiley & Sons, New York.

Joshi, P.C. (1978): Recurrence relations between moments of order statistics from exponential and truncated exponential distributions. *Sankhyā, Ser. B*, **39**, 362-371.

Joshi, P.C. (1982): A note on the mixed moments of order statistics from exponential and truncated exponential distributions. *J. Statist. Plann. Inference*, **6**, 13-16.

Kamps, U. (1991): A general recurrence relation for moments of order statistics in a class of probability distributions and characterizations. *Metrika,* **38**, 215-225.

Kamps, U. (1995): *A concept of Generalized Order Statistics*. Teubner Stuttgart.

Kamps, U. (1998): Characterizations of distributions by recurrence relations and identities for moments of order statistics. In: N. Balakrishnan and C.R. Rao, (Eds.), Handbook of Statistics, vol. 16, Order Statistics: Theory and Methods, North-Holland, Amsterdam, 291-311.

Kamps, U. and Cramer, E. (2001): On distributions of generalized order statistics. *Statistics*, **35**, 269-280.

Keseling, C. (1999): Conditional distributions of generalized order statistics and some characterizations. *Metrika*, **49**, 27-40.

Khan, A.H. and Abouammoh, A.M. (2000): Characterization of distributions by conditional expectation of order statistics. *J. Appl. Statist. Sci*., **9**, 159-167.

Khan, A.H. and Abu-Salih, M.S. (1989): Characterizations of probability distributions by conditional expectation of order statistics. *Metron*, **XLVII**, 171-181.

Khan, A.H. and Ali, M.M. (1987): Characterizations of probability distributions through higher order gap. *Comm. Statist. Theory Methods*, **16**, 1281-1287.

Khan, A.H. and Alzaid, A.A. (2004): Characterization of distributions through linear regression of non-adjacent generalized order statistics. *J. Appl. Statist. Sci.*, **13**, 123-136.

Khan, A.H. and Athar, H. (2004): Characterization of distributions through order statistics. *J. Appl. Statist. Sci.*, **13**(2), 147-154.

Khan, A.H. and Khan, I.A. (1987): Moments of order statistics from Burr distribution and its characterizations. *Metron*, **45**, 21-29.

Khan, A.H. and Khan, M.J.S. (2009): On characterization of continuous distributions conditioned on a pair of non-adjacent records. *J. Appl. Probab. Statist.*, **4**(1), 65-75.

Khan, A.H., Athar, H. and Chishti, S. (2007b): On characterization of continuous distributions conditioned on a pair of order statistics. *J. Appl. Statist. Sci.*, **16**(3), 331-346.

Khan, A.H., Anwar, Z. and Chishti, S. (2010a): Characterization of continuous distributions through conditional expectation of functions of dual generalized order statistics. *Pakistan J. Statist*., **26**(4), 615-628.

Khan, A.H., Faizan, M. and Haque, Z. (2010b): Characterization of continuous distributions through record statistics. *Commun. Korean Math. Soc.*, **25**, 485-489.

Khan, A.H., Khan R.U., and Parvez, S. (1984): Inverse moments of order statistics from Weibull distribution. *Scand. Actuar. J.*, 91-94.

Khan, A.H., Khan, R.U. and Yaqub, M. (2006): Characterization of continuous distributions through conditional expectation functions of generalized order statistics. *J. Appl. Prob. Statist.*, **1**, 115-131.

Khan, A.H., Yaqub, M. and Parvez, S. (1983a): Recurrence relations between moments of order statistics. *Naval Res. Logist. Quart.*, **30**, 419-441. *Corrections*, **32** (1985), 693.

Khan, A.H., Parvez, S. and Yaqub, M. (1983b): Recurrence relations between product moments of order statistics. *J. Statist. Plann. Inference*, **8**, 175-183.

Khan, A.H., Shah, I.A. and Ahsanullah, M. (2012): Characterization through distributional properties of dual generalized order statistics. *J. Egyptian Math. Soc.*, **20**, 211-214.

Khan, R.U. and Athar, H. (2010): Characterization of probability distributions through conditional expectation of record values. *J. Appl. Probab. Statist.*, **5**(1), 43-51.

Khan, R.U., Anwar, Z. and Athar, H. (2007a): Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution. *Aligarh J. Statist.*, **27**, 69-79.

Kim, C. and Han, K. (2010): Estimation of the scale parameter of the half-logistic distribution under progressively Type-II censored sample. *Statist. Papers*, **51**, 375-387.

Kim, C., Jung, J. and Chung, Y. (2011): Bayesian estimation for the exponentiated Weibull model under Type-II progressive censoring. *Statist. Papers*, **52**, 53-70.

Lawless, J.F. (2003): *Statistical Models and Methods for Lifetime data*. John Wiley & Sons, New York.

Lehmann, E.L. (1986): *Testing Statistical Hypothesis.* John Wiley & Sons, New York.

Lieblein, J. (1955): On moments of order statistics from the Weibull distribution. *Ann. Math. Statist.*, **26**, 330-333.

Lindley, D.V. (1958): Fiducial distributions and Bayes' theorem. *J. R. Stat. Soc. Ser. **B** Stat. Methodol.*, **20**, 102-107.

López-Blázquez, F. and Moreno-Rebollo, J.L. (1997): A characterization of distributions based on linear regression of order statistics and record values. *Sankhy $\bar{a}$, Ser. A*, **59**, 311-323.

Mahmoud, M.A.W. and Al-Nagar, H.SH. (2009): On generalized order statistics from linear exponential distribution and its characterization. *Statist. Papers*, **50**, 407-418.

Mahmoud, M.R., Sultan, K.S. and Saleh, H.M. (2006): Progressively censored data from the linear exponential distribution: moments and estimation. *Metron*, **LXIV**(2), 199-215.

Malik, H.J. (1966): Exact moments of order statistics from the Pareto distribution. *Skandinavisk Aktuarietidskrift*, **49**, 144-157.

Malik, H.J., Balakrishnan, N. and Ahmed, S.E. (1988): Recurrence relations and identities for moments of order statistics, I: Arbitrary continuous distributions. *Comm. Statist. Theory Methods*, **17**(8), 2623-2655.

Mann, N.R. (1969): Exact three-order-statistics confidence bounds on reliable life for a Weibull model with progressive censoring. *J. Amer. Statist. Assoc.*, **64**, 306-315.

Mann, N.R. (1971): Best linear invariant estimation for Weibull parameters under progressive censoring. *Technometrics*, **13**, 521-534.

Mann, N.R. and Fertig, E.W. (1973): Tables for obtaining Weibull confidence bounds and tolerance bounds based on best linear invariant estimates of parameters of the extreme-value distribution. *Technometrics*, **15**, 87-101.

Martz, H.F. and Waller, R.A. (1982): *Bayesian Reliability Analysis*. John Wiley & Sons, New York.

Mbah, A.K. and Ahsanullah, M. (2007): Some characterization of the power function distribution based on lower generalized order statistics. *Pakistan J. Statist.*, **23**(2), 139-146.

Meeker, W.Q. and Escobar, L.A. (1998): *Statistical Methods for Reliability Data*. John Wiley & Sons, New York.

Minimol, S. and Thomas, P.Y. (2013): On some properties of Makeham distribution using generalized record values and its characterization. *Braz. J. Prob. Stat.*, **27**, 487-501.

Nagaraja, H.N. (1977): On a characterization based on record values. *Austral. J. Statist.*, **19**(1), 70-73.

Nagaraja, H.N. (1988a): Record values and related statistics- a review. *Comm. Statist. Theory Methods*, **17**, 2223–2238.

Nagaraja, H.N. (1988b): Some characterizations of continuous distributions based on regressions of adjacent order statistics and record values. *Sankhyā, Ser. A*, **50**, 70-73.

Nayabuddin and Athar, H. (2016): Recurrence relations for single and product moments of generalized order statistics from Marshall-Olkin extended Pareto distributions. *Comm. Statist. Theory Methods*, DOI: 10.1080/03610926.2014.960579

Nevzorov, V.B. (2001): *Records: Mathematical Theory. Translations of Mathematical Monographs, vol. 194.* American Mathematical Society, Providence, RI.

Noor, Z. and Athar, H. (2014): Characterization of probability distributions by conditional expectation of function of record statistics. *J. Egyptian Math. Soc.*, **22**, 275-279.

Noor, Z., Akhter, Z. and Athar, H. (2015): On characterization of probability distributions through conditional expectation of generalized and dual generalized order statistics. *Pakistan J. Statist.*, **31**(2), 159-170.

Pawlas, P. and Szynal, D. (1998): Relations for single and product moments of $k-th$ record values from exponential and Gumbel distributions. *J. Appl. Statist. Sci.*, **7**(1), 53-62.

Pawlas, P. and Szynal, D. (1999): Recurrence relations for single and product moments of $k-th$ record values from Pareto, generalized Pareto and Burr distributions. *Comm. Statist. Theory Methods*, **28**(7), 1699-1709.

Pawlas, P. and Szynal, D. (2001a): Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. *Demonstratio Mathematica*, **XXXIV**(2), 353-358.

Pawlas, P. and Szynal, D. (2001b): Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. *Comm. Statist. Theory Meth.*, **30**(4), 739-746.

Raqab, M.Z. (2000): On the moments of record values. *Comm. Statist. Theory Methods*, **29**(7), 1631-1647.

Raqab, M.Z. (2001): Some results on the moments of record values from linear exponential distribution. *Math. Comput. Modelling*, **34**, l-8.

Raqab, M.Z. (2004): Generalized exponential distribution: Moments of order statistics. *Statistics*, **38**, 29-41.

Rinne, H. (2009): *The Weibull Distribution: A Handbook.* Chapman & Hall/ CRC Press, Boca Raton, FL.

Roberts, H.R. (1962a): *Some results in life testing based on hypercensored samples from an exponential distribution*, Ph.D. Dissertation, The George Washington University, Washington, DC.

Roberts, H.R. (1962b): Life test experiments with hypercensored samples. *Proceedings Eighteenth Annual Quality Control Conference, Rochester Society for Quality Control.* Rochester, New York.

Rohatgi, V.K. and Saleh, A.K.Md.E. (1988): A class of distributions connected to order statistics with nonintegral sample size. *Comm. Statist. Theory Methods.*, **17**, 2005-2012.

Samuel, P. (2008): Characterization of distributions by conditional expectation of generalized order statistics. *Statist. Papers*, **49**, 101-108.

Saran, J. and Pandey, A. (2003): Recurrence relations for marginal and joint moment generating functions of generalized order statistics from power function distribution. *Metron*, **LXI**(1), 27-33.

Saran, J. and Pushkarna, N. (2001): Recurrence relations for moments of progressive Type-II right censored order statistics from Burr distribution. *Statistics*, **35**, 495-507.

Sarhan, A.E. and Greenberg, B.G. (1962): *Contributions to order statistics.* John Wiley & Sons, New York.

Soliman, A.A. and Al-Aboud, F.M. (2008): Bayesian inference using record values from Rayleigh model with application. *European J. Oper. Res.*, **185**, 659-672.

Stigler S.M. (1977): Fractional order statistics, with applications. *J. Am. Statist. Assoc.*, **72**, 544-550.

Thomas, P.Y. and Samuel, P. (2008): Recurrence relations for the moments of order statistics from a beta distribution. *Statist. Papers*, **49**, 139-146.

Viveros, R. and Balakrishnan, N. (1994): Interval estimation of parameters of life from progressively censored data. *Technometrics*, **36**, 84-91

Weibull, W. (1939a): A statistical theory of the strength of material. *Ingeniörs vetenskaps Akademien Handligar*, Number 151, Generalstabens Litografiska Anstalts Forlag, Stockholm.

Weibull, W. (1939b): The phenomenon of rupture in solids. *Ingeniörs vetenskaps Akademien Handligar*, Number 153, Generalstabens Litografiska Anstalts Forlag, Stockholm.

Wesolowski, J. and Ahsanullah, M. (1997): On characterizing distributions via linearity of regression for order statistics. *Aust. J. Statist.*, **39**, 69-78.

Wilks, S.S. (1942): Statistical prediction with special reference to the problem of tolerance limits. *Ann. Math. Stat.*, 13, 400-409.

Wilks, S.S. (1948): Order Statistics. *Bull. Amer. Math. Soc.*, **5**, 06-50.

Wu, J.W. (2004): On characterizing distributions by conditional expectation of functions of non-adjacent record values. *J. Appl. Statist. Sci.*, **13**, 137-145.

Wu, J.W. and Lee, W.C. (2001): On characterization of generalized extreme value, power function, generalized Pareto and classical Pareto distributions by conditional expectation of record values. *Statist. Papers*, **42**, 225-242.

Wu, J.W. and Ouyang, L.Y. (1996): On characterizing distributions by conditional expectations of functions of order statistics. *Metrika*, **43**, 135-147.

Yanev, G.P., Ahsanullah, M. and Beg, M.I. (2008): Characterizations of probability distributions via bivariate regression of record values. *Metrika*, **68**, 51-64.